CLASSICAL MECHANICS

M.Sc. PHYSICS

SEMESTER-I, PAPER-I

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FOREWORD

Since its establishment in 1976, Acharya Nagarjuna University has been forging ahead in the path of progress and dynamism, offering a variety of courses and research contributions. I am extremely happy that by gaining 'A+' grade from the NAAC in the year 2024, Acharya Nagarjuna University is offering educational opportunities at the UG, PG levels apart from research degrees to students from over 221 affiliated colleges spread over the two districts of Guntur and Prakasam.

The University has also started the Centre for Distance Education in 2003-04 with the aim of taking higher education to the door step of all the sectors of the society. The centre will be a great help to those who cannot join in colleges, those who cannot afford the exorbitant fees as regular students, and even to housewives desirous of pursuing higher studies. Acharya Nagarjuna University has started offering B.Sc., B.A., B.B.A., and B.Com courses at the Degree level and M.A., M.Com., M.Sc., M.B.A., and L.L.M., courses at the PG level from the academic year 2003-2004 onwards.

To facilitate easier understanding by students studying through the distance mode, these self-instruction materials have been prepared by eminent and experienced teachers. The lessons have been drafted with great care and expertise in the stipulated time by these teachers. Constructive ideas and scholarly suggestions are welcome from students and teachers involved respectively. Such ideas will be incorporated for the greater efficacy of this distance mode of education. For clarification of doubts and feedback, weekly classes and contact classes will be arranged at the UG and PG levels respectively.

It is my aim that students getting higher education through the Centre for Distance Education should improve their qualification, have better employment opportunities and in turn be part of country's progress. It is my fond desire that in the years to come, the Centre for Distance Education will go from strength to strength in the form of new courses and by catering to larger number of people. My congratulations to all the Directors, Academic Coordinators, Editors and Lesson-writers of the Centre who have helped in these endeavors.

Prof. K. Gangadhara Rao
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M.SC. PHYSICS
SYLLABUS
SEMESTER-I, PAPER-I
101PH24-CLASSICAL MECHANICS

Course Objectives:

- ❖ Introduction to basic ideas about Newtonian mechanics
- ❖ Initiation of mechanical system through derivative and problematic approaches
- ❖ Study of motion of the body in different systems of equation

Unit-I (Lagrangian Mechanics):

Newtonian mechanics of one and many particle systems, Conservation laws, Constraints and their classification, principle of virtual work, D'Alembert's principle and Lagrange's equation of motion, Applications: linear harmonic oscillator, simple pendulum, compound pendulum, L-C Circuit, Lagrangian for a Charged Particle Moving in an Electromagnetic field.

Learning Outcomes:

- Learning concepts of mechanics of the systems for problematic analysis of the objects
- Lagrangian systems are useful to examination of the motion of the objects
- In view of Competitive exams problematic and derivational tactics in equation of motion in Lagrangian from D'Alembert's principle.

Unit-II (Hamilton's Mechanics):

Deduction of Hamilton's principle from D'Alemberts principle, modified Hamilton's principle, Hamilton's principle and Lagrange's equations, generalized momentum and cyclic coordinates, Hamilton function H and conservation of energy, Hamilton's equation (Hamilton's canonical equations of motion), Simple application of the Hamilton principle-linear harmonic oscillator, simple pendulum, A-variation, principle of least action. Equationsofcanonical transformation, (Generating functions), examples of canonical transformations for a harmonic oscillator.

Learning Outcomes:

- To study the Hamilton's principle from D'Alemberts principle.
- To learn about oscillator mechanics and canonical transformations.

Unit-III (Poisson's Bracket and Hamilton-Jacobi Method):

Introduction to Poisson's bracket notation, equations of motion in Poisson bracket form, fundamentals of Poisson's bracket notation, angular momentum and Poisson brackets, Jacobi's identity.

Hamilton-Jacobi equation of Hamilton's principal function, The Harmonic oscillator problem as an example of the Hamilton-Jacobi Method, Hamilton-Jacobi equation for Hamilton's characteristic function, Action-angle variables.

Learning Outcomes:

- To study the equation of motion in Poisson bracket form
- In view of theory exams theory learning for Hamilton's-Jacobi equations.
- Learn about Hamilton-Jacobi equation for Hamilton's characteristic function.

Unit-IV (Dynamics of a Rigid Body):

The Euler angles-first rotation, second rotation and third rotation, angular momentum and inertia tensor, principal axes and principal moments of inertia, rotational kinetic energy of a rigid body, Euler's equations of motion of a rigid body, torque-free motion of a rigid body.

Learning Outcomes:

- Gained knowledge of The Euler angles-first rotation, second rotation and third rotation.
- Learn about motion and indication of rigid body through tensor, Euler equation of motion.

Unit-V (Special Theory of Relativity):

Introduction to special theory of relativity, Galilean transformations, principle of relativity, transformation of force from one inertial system to another, covariance of the physical laws, principle of relativity and speed of light, Lorentz transformations, consequences of Lorentz transformations, aberration of light from stars, relativistic Doppler's effect.

Learning Outcomes:

- Galilean transformations of relativistic mechanics.
- Covariance of the physical laws
- Relativistic Doppler's effect.

Course Outcomes:

- Students get knowledge on mechanics of the macroscopic things where Newton laws are applicable, can learn constrained motion of rigid bodies in one, two and three dimensions.
- Students can understand the motion of bodies similar to Hamilton and Lagrangian systems and resolve with practical approach.
- The students will know the concept of special theory of relativity.

Text and Reference Books:

1) Classical Mechanics by H.Goldstein
2) Fundamentals of Classical Mechanics by J.C. Upadhyaya,
3) Classical Mechanics by Charles P.Poole, John Safko 3rd Edition, Parson Publications
4) Classical Mechanics by G. Aruldhas, PHI Publishers
5) Introduction to special relativity- Robert Resnick.

(101PH24)

M.Sc. DEGREE EXAMINATION, MODEL QUESTION PAPER
M.Sc. PHYSICS-FIRST SEMESTER
CLASSICAL MECHANICS

Time: Three hours **Maximum: 70 marks**

Answer ALL Questions

All Questions Carry Equal Marks

1 a) What is D'Alembert's principle? Derive Lagrange's equation from D'Alembert's principle

b) Derive Lagrange's equation from Hamilton's principle.

OR

c) Derive Hamilton equations motion. Show a simple pendulum as an application

d) State Lagrange-Brackets and their applications.

2 a) What is Hamilton Jacobi equation?

b) Define Angular momentum and torque. Write a note on the inverse square law of forces.

OR

c) What is the Doppler effect? What are the applications of relativistic dynamics of a single particle?

d) Discuss Kepler's problem in action-angle variables.

3 a) Write a brief note on rigid body dynamics.

b) Discuss Eulerian angles. Write a brief on Euler's equation of a rigid body

OR

c) Transformations for the acceleration of a particle.

d) Write a note on the transformation of momentum and force.

4 a) Describe the experimental verification of the variation of mass with velocity

b) What is the condition for the transformation to be canonical?

OR

c) Discuss constraints and their classifications.

d) Discuss the canonical transformations in detail and explain the condition for a transformation to be canonical

5 a) Show linear harmonic oscillator as a simple application of the Hamilton principle. Derive simple pendulum, Δ-variation

b) Derive Lagrange's Equation from Hamilton's Principle. Write modified Hamilton's principle.

OR

c) Write the Hamilton's principle and Lagrange's equations. Discuss about velocity-dependent potential.

d) What are Lagrangian applications? Prove the laws of conservation of linear momentum, angular momentum and energy for a system of particles.

CONTENTS

LESSON-1

NEWTONIAN MECHANICS

1.0 AIM AND OBJECTIVES:

To provide a set of fundamental principles that describes the relationship between the motion of an object and the forces acting upon it. To create a framework for predicting and explaining the motion of objects in the macroscopic world. To define inertia and explain that objects resist changes in their state of motion. To establish that an object will remain at rest or in uniform motion unless acted upon by a net external force. To quantify the relationship between force, mass, and acceleration (F=ma). To provide a means of calculating the acceleration of an object when subjected to a given force. To establish that forces always occur in pairs. To explain that for every action, there is an equal and opposite reaction. To develop a systematic approach to analyzing the motion of individual particles under the influence of forces. To provide the tools and methods for understanding and predicting the trajectory and behaviour of particles. To apply Newton's laws to solve problems involving the motion of particles. To analyze the effects of various forces (gravity, friction, etc.) on particle motion. To understand concepts such as displacement, velocity, acceleration, and momentum in the context of particle motion. To allow the ability to use vector analysis to describe the movement of particles in 3 dimensional space accurately. To identify and understand fundamental quantities that remain constant in physical systems. To provide powerful tools for analyzing and solving problems in mechanics and other areas of physics. To establish that the total momentum of a closed system remains constant. To use this principle to analyze collisions and other interactions between objects. To establish that the total energy of a closed system remains constant. To understand the transformations between different forms of energy (kinetic, potential, etc.). To establish that the total angular momentum of a closed system remains constant. To analyze rotational motion and understand the factors that affect it. In essence, these principles and laws work together to provide a comprehensive framework for understanding and predicting the motion of objects in the physical world.

STRUCTURE:

1.1 NEWTON'S LAWS OF MOTION:

Sir Isaac Newton expressed his ideas regarding the motion of bodies in the form of three laws which are considered as the basic laws of mechanics. In fact mechanics is a study of certain general relations describe the interactions of material bodies. One general property of a material body is its inertial mass. Another new concept useful in describing interactions is force. These two concepts, inertial mass and force were first defined in a quantitative manner by Isaac Newton. The definition of mass and force are containing his three laws of motion.

Law of Inertia (First Law): A body continues in its state of rest or constant velocity, unless disturbed by some external influence. The property of a body that it cannot change its state of rest or constant velocity is called inertia and the influence under which the velocity of a particle changes is called force. Quantitative definitions of force and measure of inertia of a body, which we call mass are contained in second and third laws of motion.

Law of Force (Second Law): The time-rate of change of momentum is proportional to the impressed force, i.e.,

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} \tag{1}$$

Everybody possesses the property of inertia or resistance to motion. This inertia is different for different bodies. The measure to this inertia for translation is called the mass of a body and is denoted by m. If **v** be the velocity of a body of mass m, then its momentum is defined by

$$\mathbf{p} = m\mathbf{v} \text{ and thus } \boldsymbol{F} = \frac{d}{dt}(m\mathbf{v}) \tag{1a}$$

Newton considered that mass of a body remains constant in motion. Therefore,

$$\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{a} \tag{1b}$$

i.e. Force = mass x acceleration

This is the fundamental law of classical mechanics. Quantitatively, first law is the special case of second law, because if force is not acting on a body, i.e., F = 0, then $\frac{dv}{dt} = 0$and therefore v = constant, including zero.

Law of Action and Reaction (Third Law): To every action there is always equal and opposite reaction. This means that if 1 and 2 bodies are interacting mutually, then

$$\mathbf{F}_{12} = -\mathbf{F}_{21} \tag{2}$$

i.e., force on 1st body due to 2nd = - force on 2nd body due to 1st.

1.2 MECHANICS OF A PARTICLE: CONSERVATION LAWS

Conservation of Linear Momentum: If a force F acting on a particle of mass m, then according to Newton's law of motion, we have

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = \frac{d}{dt}(m\boldsymbol{v}) \qquad (3)$$

where p = mv is the linear momentum of particle.

If the external force acting on the particle is zero, then

$$\frac{d\mathrm{p}}{dt} = \frac{d}{dt}(m\mathbf{v}) = 0$$

or $$\mathbf{p} = m\mathbf{v} = \text{constant} \qquad (4)$$

Thus, in the absence of external force, the linear momentum of a particle is conserved. This is the conservation theorem for a free particle.

Conservation of Angular Momentum: The angular momentum of a particle P of mass m about a point O is defined as

$$\mathbf{J} = \mathbf{r} \times \mathbf{p} \qquad (5)$$

where $\mathbf{r}$ is the position vector of the particle P and $\mathbf{p}$ = m$\mathbf{v}$ is its linear momentum.If the force on the particle is **F**, then the moment of force or torque about O is defined as

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \qquad (6)$$

By differentiating (1.3) with respect to t, then

$$\frac{dJ}{dt} = \frac{d}{dt}(r \times p) = \frac{dr}{dt} \times p + r \times \frac{dp}{dt}$$

or $\frac{dJ}{dt} = r \times F \left[\because \frac{dr}{dt} \times p = v \times mv = 0\right]$

Therefore, $\boldsymbol{\tau} = \frac{d\mathbf{J}}{dt} = \frac{d}{dt}(\mathbf{r} \times \mathbf{p}) \qquad (7)$

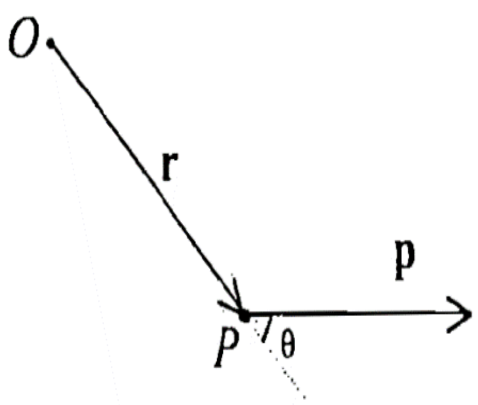

Fig. 1.1: Angular Momentum of a Particle P along a Point O.

Thus, the time rate of change of angular momentum of a particle is equal to the torque acting on it. This analogues to the equation (3) for linear motion.

If the torque acting on a particle is zero, then

$$\boldsymbol{\tau} = \frac{d\mathbf{J}}{dt} = 0 \; or \; \mathbf{J} = \text{constant} \qquad (8)$$

Therefore, angular momentum of a particle is constant of motion in absence of external torque.

Conservation of Energy:

Work: Work done by an external force F upon a particle in displacing from point 1 to another point 2 is defined as

$$W_{12} = \int_1^2 \mathbf{F}.d\mathbf{r} \qquad (9)$$

Kinetic Energy and Work-Energy theorem:

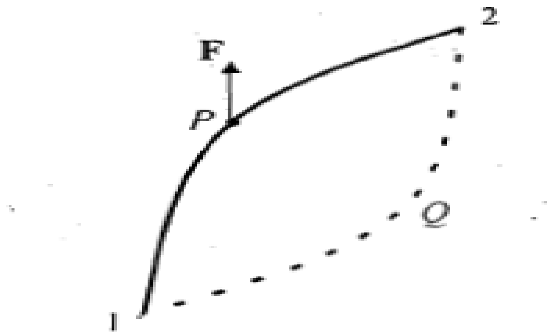

Fig. 1.2: Workdone by a Force on a Particle

According to newton's 2^{nd} law **F** = m d**v**/dt and hence

$$\mathbf{F}.d\mathbf{r} = m\frac{d\mathbf{v}}{dt}.d\mathbf{r} = m\frac{d\mathbf{v}}{dt}.\mathbf{v}\,dt \left[\because d\mathbf{r} = \frac{d\mathbf{r}}{dt}dt = \mathbf{v}\,dt\right]$$

$$= \text{m}\frac{d}{dt}\left[\frac{1}{2}\mathbf{v}.\mathbf{v}\right]\text{dt} = d\left[\frac{1}{2}\text{mv}^2\right]$$

Therefore, equation (7) is obtained as

$$W_{12} = \int_1^2 F.dr = T_2 - T_1 \qquad (10)$$

This is known as work-energy theorem.

Conservation of Force and Potential energy: If the work done (W_{12}) by force in moving a particle from point 1 to point 2 is the same for any possible path between the points, then the force is said to be conservative. The region in which the particle is experiencing a conservative force is called as conservative force field.

Thus, for conservative force (Fig. 1.2)

$$P\int_1^2 \mathrm{F}.d\mathrm{r} = Q\int_1^2 \mathrm{F}.d\mathrm{r} \;\; or \;\; P\int_1^2 \mathrm{F}.d\mathrm{r} + Q\int_2^1 \mathrm{F}.d\mathbf{r} = 0 \;\; i.e., \oint F.dr = 0 \quad (11)$$

Thus, if the force is conservative, the work done on the particle around a closed path in the force field is zero. In case of a non-conservative force like friction, the amount of work done around different closed paths are different and not zero.

According to stokes theorem,

$$\oint \boldsymbol{F}.d\boldsymbol{r} = \iint curl\, \boldsymbol{F}.d\boldsymbol{s}$$

Since the work done around the closed path is zero, it does not depend on its length. So, we may do integration over the perimeter of the area d**s**.

$$\oint \mathbf{F}.d\mathbf{r} = \mathrm{curl}\, \mathbf{F}.d\mathbf{s} = 0$$

But, ds ≠ 0 and hence in general

$$\mathrm{curl}\ \mathbf{F} = 0 \text{ or } \boldsymbol{\nabla} \times \mathbf{F} = 0 \quad (12)$$

Therefore, force can be expressed as

$$\mathbf{F} = -\boldsymbol{\nabla} V = -\left(\hat{\mathbf{i}}\frac{\partial V}{\partial x} + \hat{\jmath}\frac{\partial V}{\partial y} + \hat{k}\frac{\partial V}{\partial z}\right) \quad (13)$$

Because $\nabla \times \nabla V = \hat{\mathbf{i}}\left(\frac{\partial^2 V}{\partial y \partial z} - \frac{\partial^2 V}{\partial z \partial y}\right) + \hat{\mathbf{j}}\left(\frac{\partial^2 V}{\partial z \partial x} - \frac{\partial^2 V}{\partial x \partial z}\right) + \hat{\mathbf{k}}\left(\frac{\partial^2 V}{\partial x \partial y} - \frac{\partial^2 V}{\partial y \partial x}\right) = 0$

This scalar function V is known as potential or potential energy and depends on position.

$$\int_1^2 \boldsymbol{F}.d\boldsymbol{r} = -\int_1^2 \nabla V.d\boldsymbol{r} = -\int_1^2 \mathrm{dV} = V_1 - V_2 \quad (14)$$

Now, if we assume the position 1 is at infinity and the potential at infinity is zero.

Now potential at a point **r** is given by

$$V(r) = -\int_\infty^r \mathbf{F}.d\mathbf{r} \quad (15)$$

From equation (1.14), the work done by the conservative force is

$$W_{12} = \int_1^2 \mathbf{F}.d\mathbf{r} = V_1 - V_2 \quad (16)$$

This gives the change in potential when particle moved from position 1 to position 2.

Conservation Theorem: According to equation (10), the amount of work done by a force in moving a particle from position "1" to "2" is given by the change in kinetic energy i.e.,

$$W_{12} = \int_1^2 \mathbf{F}.d\mathbf{r} = T_2 - T_1 \quad (17)$$

Therefore, from eq. (16) and (17),

$$V_1 - V_2 = T_2 - T_1 \text{ or } T_1 + V_1 = T_2 + V_2 = \text{constant} \quad (18)$$

Thus, the sum of kinetic and potential energies (i.e., total mechanical energy) of a particle remains constant in a conservative force field. This is known as the law of conservation of energy.

Remember that the law of conservation of energy gives us no new information, not contained in Newton's second law of motion. If we multiply by **v** = d**r**/dt to both sides of **F** = m. d**v**/dt and integrate with respect to t, we obtain

$$\int \mathrm{m}\frac{\mathrm{d}\mathbf{v}}{\mathrm{dt}}.\mathbf{v}dt = \int \mathbf{F}.\frac{\mathrm{d}\mathbf{r}}{\mathrm{dt}}dt + \text{constant (say E)}$$

$$\int \frac{d}{dt}\left[\frac{1}{2}m\mathbf{v}\,.\,\mathbf{v}\right]dt = \int \mathbf{F}.\mathrm{d}\mathbf{r} + E$$

$$\int d\left[\frac{1}{2}m\mathrm{v}^2\right] - \int \mathbf{F}.\mathrm{d}\mathbf{r} = E \; or \; \frac{1}{2}m\mathrm{v}^2 - \int_{\infty}^{r}\mathbf{F}.\mathrm{d}\mathbf{r} = E$$

i.e. $$\mathrm{T + V = E} \qquad (19)$$

where the constant E is the total energy of the particle. The above Equation represents the conservation energy theorem.

Conservation laws, obtained above, are the constants of motion and referred as the first integrals of the motion. They are very useful because we get some important information physically about the system just at a glance from these integrals. In fact once integration of the equation of motion under certain condition on the system yields the first integral. Since Newton's equation is a second order differential equation, these first integrals of motion are in fact first order differential equations.

1.3 MECHANICS OF A SYSTEM OF PARTICLES:

1.3.1 External and Internal Forces

In the last section, we arrived at some results, especially conservation theorems, for the mechanics of a particle. These results can be easily generalized to the case of a system of particles by taking care of mutual interactions. Now, if a mechanical system consists of two or more particles, then the force on the ith particle is given by

$$F_i = F_i^e + \sum_{j=1}^{N} F_{ij} \qquad (20)$$

where F_i^e is the external force, acting on the ith particle due to sources outside the system. $F_{ij}$is the internal force on the ith particle due to the jth particle and the total internal force due to all other particles (j=l to N) on the ith particle is represented by the sum in equation (20), where N is the number of particles in the system and F_{ii}, the force of ith particle on itself, is naturally zero.

According to Newton's second law

$$F_i = \dot{p}_i = m_i \frac{dv_i}{dt} = m_i \frac{d^2 r_i}{dt^2}$$

Now, when the sum is taken over all the particles of the system, equation (20) takes the form

$$\frac{d^2}{dt^2} \sum_i m_i \mathrm{r}_i = \sum_i F_i^e + \sum_i \sum_j F_{ij} \qquad (i \neq j) \qquad (21)$$

On the right hand side of equation (21) first sum represents the total external force F^e. According Newton's third law, any two particles of the system exert equal and opposite forces on each other, i.e.,

$$F_{ij} = F_{ji} \qquad (22)$$

Since the second sum in equation (21) represents the internal forces in pairs and for each pairresultant force is zero, consequently this sum vanishes.

Thus, equation (21) is

$$F^e = \frac{d^2}{dt^2} \sum_i m_i \mathrm{r}_i \qquad (23)$$

1.3.2 Centre of Mass:

We define the centre of mass R of the system by

$$R = \frac{\sum_i m_i \mathrm{r}_i}{\sum_i m_i} = \frac{\sum_i m_i \mathrm{r}_i}{M} \qquad (24)$$

where $\sum_i m_i$ = M is the total mass of the system. In view of eq. (23), eq. (24) assumes the form

$$F^e = M \frac{d^2 \boldsymbol{R}}{dt^2} = M \mathrm{a}_{cm} \qquad (25)$$

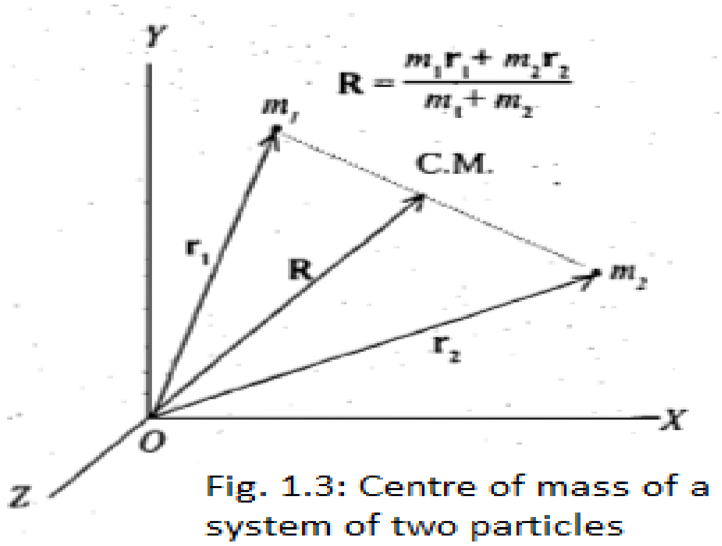

Fig. 1.3: Centre of mass of a system of two particles

Thus, the acceleration of the centre of mass is due to only the external forces and is given by Newton's second law of motion. Thus, the centre of mass of a system of particles moves as if it were a particle of mass equal to the total mass of the system subjected to the external forces applied on the Z System

1.3.3 Conservation of Linear Momentum:

If we differentiate eq. (1.24) with respect to t, we obtain

$$M\frac{dR}{dt} = m_1\frac{dr_1}{dt} + m_2\frac{dr_2}{dt} + \cdots + m_N\frac{dr_N}{dt}$$

or
$$MV = m_1v_1 + m_2v_2 + \cdots + m_Nv_N = \sum_{i=1}^{N} m_i\mathrm{v}_i \qquad (26)$$

which gives the velocity (V) of center of mass. The sum $\sum m_i\mathrm{v}_i = \mathbf{P}$ is the total linear momentum of all the particles of the system

Thus P=MV (27)

Thus, the total linear momentum of the system is equal to the product of total mass of the system and the velocity of Centre of mass.

Differentiating eq. (27) with respect to , we get

$$\frac{d\mathbf{P}}{dt} = \frac{d(MV)}{dt} = M\frac{dV}{dt} + M\frac{d^2R}{dt^2} \qquad (28)$$

Hence by using eq. (25), the total external force on the system is

$$F^e = \frac{d\mathbf{P}}{dt} = \frac{d(MV)}{dt} \qquad (29)$$

When $F^e = 0$,

$$P = MV = \sum_i m_i\mathrm{v}_i \qquad (1.30)$$

Thus, if the total external force F on the system is zero, its total linear momentum is the constant ofmotion. This is the law of conservation of linear momentum for a system.

1.3.4 CENTRE OF MASS-FRAME OF REFERENCE:

An inertial frame attached with the centre of mass of an isolated system (i.e., a system free from external forces) of particles is called the centre of mass-frame of reference or C-frame of reference. In this. C-frame of reference, the centre of mass remains at rest i.e., V= 0. So that in view of eq(27), the total linear momentum of the system in C-frame of reference is always zero, i.e.

$$P = MV = \sum_i m_i\mathrm{v}_i = 0 \text{ (in C} - \text{frame of reference)}$$

This is why the C-frame is called the zero-momentum frame. This C-frame is important because several experiments which we perform in the laboratory (or L-frame) can be more simply analyzed in the centre of mass frame of reference.

1.3.5 CONSERVATION OF ANGULAR MOMENTUM:

If $\mathbf{J_1}$, $\mathbf{J_2}$ are the angular momenta of various particles of a system about a given point O, the total angular momentum about the point O is given by

$$\mathbf{J} = \mathbf{J_1} + \mathbf{J_2} + \cdots + \mathbf{J_N} = (\mathbf{r_1} \times \mathbf{p_1}) + (\mathbf{r_2} \times \mathbf{p_2}) + \cdots + (\mathbf{r_N} \times \mathbf{p_N})$$

$$\mathbf{J} = \sum_{i=1}^{N} (\boldsymbol{r_i} \times p_i) \quad (31)$$

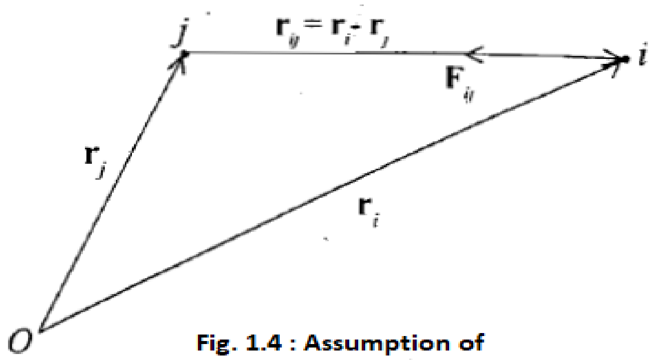

Fig. 1.4 : Assumption of the force F_{ij} parallel to r_{ij}

Also

$$\frac{d\mathbf{J}}{dt} = \sum_i (r_i \times \dot{p}_i) + \sum_i (r_i \times F_i) \qquad (\because\ \dot{r} \times p_i = v_i \times m v_i = 0) \qquad (32)$$

If we take product with r_i in eq. (32) and sum over all the particles of the system, then

$$\sum_i (r_i \times F_i) = \sum_i (r_i \times F_i^e) + \sum_i \sum_j (r_i \times F_{ij}) \qquad (33)$$

The last term contains the double sum for i, j = 1 to N and hence it is a sum of the pairs of the form, given by

$$r_i \times F_{ij} + r_j \times F_{ji} = (r_i - r_j) \times F_{ij} = r_{ij} \times F_{ij}$$

because $F_{ji} = -F_{ij}$ according to Newton's third law of motion

Now, if the internal forces between any two particles of the system in addition to being equal and opposite be central i.e., lie along the line joining them, then from the property of cross product $r_i \times F_{ij} = 0$.

Thus, the last term of eq. (33) vanishes and hence

$$\sum_i (r_i \times F_i) = \sum_i (r_i \times F_i^e) = \tau^e$$

But from equation (32), we have

$$\sum_i (r_i \times F_i) = \frac{d\mathrm{J}}{dt}$$

Thus, $$\tau^e = \frac{d\mathbf{J}}{dt} \qquad (34)$$

This means that the time rate of change of total angular momentum of a system of particles is equal to the applied external torque on the system about the same point.

If $\tau^e = 0$,

$$\mathbf{J} = \mathbf{J_1} + \mathbf{J_2} + \cdots + \mathbf{J_N} = \text{constant} \qquad (35)$$

In absence of the external torque, the total angular momentum of a system of particles is conserved. This is the conservation theorem for total angular momentum.

1.3.6 NOTE ON CONSERVATION THEOREMS OF LINEAR AND ANGULAR MOMENTUM FOR A SYSTEM OF PARTICLES:

We have stated the conservation theorems of linear and angular momentum of a system of particles by assuming the validity of Newton's third law for internal forces in the former case and in the later case additionally the central character of internal forces. Both of these conditions are satisfied for some physical forces, for example gravitational forces in a system, action reaction forces in a rotating mass attached to the string etc. However, there are action and reaction forces which do not obey the third law and also do not lie along the line joining the two particles. For example, if we consider two charges, moving with uniform velocities parallel to each other (which are not perpendicular to the line joining the two charges), they are according to Bio-Savart law, the forces on the two charges due to each other are of course equal and opposite, but they do not lie along the line joining them. Further let us consider two charges so that instantaneously one charge is moving directly towards the other but the other is moving at right angles the direction of the motion of the first. Consequently, the other charge exerts a definite force on the first charge, but it does not experience any reaction force at all. In such cases, the conservation theorems of linear and angular momentum appear not to be correct. However, the laws of linear and angular momentum a known as the fundamental laws of nature and therefore, one has to investigate for finding the way for the validity of the conservation theorems. For examples, the sum of mechanical angular momentum and electromagnetic angular momentum of a system of moving charges remains constant in time.

1.3.7 CONSERVATION OF ENERGY:

Similar to a single particle, the total amount of work done by the forces acting on various particles of the system from an initial configuration 1 to final configuration 2 is given by

$$W_{12} = \sum_{i=1}^{N} \int_1^2 F_i . dr_i = \sum_i \int_1^2 F_i^e . dr_i = \sum_i \sum_j \int_1^2 F_{ij} . dr_i \qquad (36)$$

Kinetic Energy: But according to second law,

$$F_i = m_i \frac{dv_i}{dt}$$

$$W_{12} = \sum_{i=1}^{N} \int_1^2 F_i . dr_i = \sum_i \int_1^2 m_i \dot{v}_i . v_i dt = \sum_{i=1}^{N} \int_1^2 d\left(\frac{1}{2} m_i v_i^2\right) = \left[\sum_i \frac{1}{2} m_i v_i^2\right]_1^2$$

$$W_{12} = T_2 - T_1 \qquad (37)$$

Thus, the work done is again equal to the change in kinetic energy (work-energy theorem), where

$$T = \sum_i \tfrac{1}{2} m_i v_i^2 \qquad (38)$$

denotes the kinetic energy of the system

If $\mathrm{v}_{iC} = \mathrm{v}_i \mathrm{V}$ is the velocity of the i[th] particle relative to the velocity of the centre of mass, then

$$T = \sum_i \frac{1}{2} m_i \mathrm{v_i} . \mathrm{v_i} = \sum_i \frac{1}{2} m_i (\mathrm{v_{iC}} + V) . (\mathrm{v}_{iC} + V)$$

$$\sum_i \frac{1}{2} m_i \mathrm{v}_{\mathrm{iC}}^2 + \sum_i \frac{1}{2} m_i V^2 + \sum_i m_i \mathrm{v_{iC}} . V = \sum_i \frac{1}{2} m_i \mathrm{v}_{\mathrm{iC}}^2 + \frac{V^2}{2} \sum_i m_i + V . \sum_i m_i \mathrm{v_{iC}}$$

But $\sum_i m_i = M$, total mass of the system and $\sum_i m_i \mathrm{v_{iC}} = 0$

Thus $$T = \sum_i \tfrac{1}{2} m_i \mathrm{v}_{\mathrm{iC}}^2 + \tfrac{1}{2} M V^2 \qquad (39)$$

Thus, the total kinetic energy of a system of particle is the sum of kinetic energy of motion about centre of mass plus the kinetic energy of motion of the total mass of the system, as if it were concentrated at the centre of mass.

Potential Energy: In eq. (1.36), if the external and internal forces both are conservative derivable from scalar potential, then

$$F_i^e = -\nabla_i V_i = -\left[\hat{\imath} \frac{\partial V_i}{\partial x_i} + \hat{\jmath} \frac{\partial V_i}{\partial y_i} + \hat{k} \frac{\partial V_i}{\partial z_i}\right] \qquad (40)$$

and $$F_{ij} = -\nabla_i V_{ij} = -\left[\hat{\imath} \frac{\partial V_{ij}}{\partial x_i} + \hat{\jmath} \frac{\partial V_{ij}}{\partial y_i} + \hat{k} \frac{\partial V_{ij}}{\partial z_i}\right] \qquad (41)$$

If the internal forces are central in nature, the potential energy V_{ij} will be a function of scalar distance $r_{ij} = |r_i - r_j|$ only. Then

$$V_{ij} = V_{ij}(|r_i - r_j|) \qquad (42)$$

So that $$\frac{\partial V_{ij}}{\partial x_i} = \frac{\partial V_{ij}}{\partial r_{ij}} \frac{\partial r_{ij}}{\partial x_i} = \frac{(x_i - x_j)}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}} \qquad (43)$$

Because $$r_{ij} = |r_i - r_j| = \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2\right]^{\frac{1}{2}}$$

and hence $$\frac{\partial r_{ij}}{\partial x_i} = \frac{(x_i - x_j)}{r_{ij}}$$

Similarly, $\frac{\partial V_{ij}}{\partial y_i} = \frac{(y_i - y_j)}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}}$ or $\frac{\partial V_{ij}}{\partial z_i} = \frac{(z_i - zy_j)}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}}$

Therefore, from equation (41), we have

$$F_{ij} = -\frac{1}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}} \left[(x_i - x_j)\hat{\imath} + (y_i - y_j)\hat{\jmath} + (z_i - z_j)\hat{k} \right]$$

$$= -(r_i - r_j) \frac{1}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}}$$

Also similarly, $F_{ji} = -\nabla_j V_{ij} = -\left[\hat{\imath} \frac{\partial V_{ij}}{\partial x_j} + \hat{\jmath} \frac{\partial V_{ij}}{\partial y_j} + \hat{k} \frac{\partial V_{ij}}{\partial z_j} \right]$

$$= (r_i - r_j) \frac{1}{r_{ij}} \frac{\partial V_{ij}}{\partial r_{ij}} \left[\text{Here,} \frac{\partial r_{ij}}{\partial x_i} = \frac{(x_i - x_j)}{r_{ij}} \text{etc} \right] \quad (44)$$

Thus, the internal forces F_{ij} and F_{ji} between the i[th] and j[th] particles are equal and opposite and automatically satisfy third law and lie along the line $(r_i - r_j)$ joining the two particles. Now, if we consider the last term of equation (36), then it can be written as

$$\sum_i \sum_{\substack{j \\ i \neq j}} \int_1^2 F_{ij}.dr_i = \frac{1}{2} \sum_{\substack{i \\ i \neq j}} \sum_j \int_1^2 (F_{ij}.dr_i + F_{ji}.dr_j) = \frac{1}{2} \sum_{\substack{i \\ i \neq j}} \sum_j \int_1^2 -(\nabla_i V_{ij}.dr_i + \nabla_j V_{ij}.dr_j)$$

$$= -\frac{1}{2} \sum_{\substack{i \\ i \neq j}} \sum_j \int_1^2 \nabla_{ij} V_{ij}.dr_{ij} \quad (45)$$

Because $\nabla_i V_{ij} = \nabla_{ij} V_{ij} = -\nabla_j V_{ij} \left[\because \frac{\partial V_{ij}}{\partial x_i} = \frac{\partial V_{ij}}{\partial x_{ij}} \right]$ and $dr_i - dr_j = dr_{ij}$

Thus eq. (36) in view of eqs. (40) and (45) is

$$W_{12} = -\sum_i \int_1^2 \nabla V_i.dr_i - \frac{1}{2} \sum_{\substack{i \\ i \neq j}} \sum_j \int_1^2 \nabla_{ij} V_{ij}.dr_{ij} = -\sum_i \int_1^2 dV_i - \frac{1}{2} \sum_{\substack{i \\ i \neq j}} \sum_j \int_1^2 dV_{ij}$$

$$= -\left[\sum_i [V_i]_1^2 + \sum_{\substack{i \\ i \neq j}} \sum_j [V_{ij}]_1^2 \right] = V_1 - V_2 \quad (46)$$

where V the total potential energy of the system is defined as

$$V = \sum_i V_i + \frac{1}{2} \sum_{\substack{i \\ i \neq j}} \sum_j \int_1^2 V_{ij} \quad (47)$$

Conservation Theorem: Now, we obtain from eqs. (37) and (46)

$$T_2 - T_1 = V_1 - V_2 \;\; or \; T_1 + V_1 = T_2 + V_2 \quad (48)$$

This is the law of conservation of energy for a system of particles

It is to be noted that in eq. (47) the total potential energy V has been defined, provided the external and internal forces are both derivable from scalar potentials. We may call the second term in eq. (47) as the potential energy which may not be zero and vary with time.

However, for a rigid body, the internal potential energy will remain constant. In fact, a rigid body is a system of particles with fixed inter-particle distances and therefore, the internal forces in a rigid body do not do any work, when the body moves from one configuration to another. Thus, the internal potential energy of a rigid body is constant and can be taken as zero to discuss its motion.

1.4 SUMMARY:

This lesson provides a comprehensive list of summaries related to Newton's Laws, Particle Mechanics, and Conservation Laws.

These principles and laws collectively summarizes:

- **Establish a fundamental framework:** To describe and predict the motion of objects, from individual particles to macroscopic systems, by defining the relationships between force, mass, and motion.
- **Analyze and solve motion problems:** Provide tools to calculate trajectories, understand forces like gravity and friction, and apply vector analysis in 3D space.
- **Identify and utilize conserved quantities:** To simplify complex problems by recognizing that momentum, energy, and angular momentum remain constant in closed systems, aiding in the analysis of collisions, energy transformations, and rotational motion.
- Essentially, these principles are the foundation of classical mechanics, enabling us to understand and predict how things move in the physical world.

1.5 TECHNICAL TERMS:

Angular momentum, linear momentum, total energy (T+V).

1.6 SELF-ASSESSMENT QUESTIONS:

1) Prove the laws of conservation of linear momentum, angular momentum and energy for a system of particles.
2) State and prove a work-energy theorem.

1.7 SUGGESTED READINGS:

1) Classical Mechanics: H.Goldstein
2) Mechanics: Simon
3) Mechanics: Gupta, Kumar and Sharma

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LESSON-2

LAGRANGIAN MECHANICS

2.0 AIM AND OBJECTIVES:

The motto of this lesson is to get information regarding the concepts of D'Alembert's Principle from the Lagrangian equation and to understand and derivate the concepts of constraints and Virtual work. To define and mathematically represent limitations on the possible motions of a mechanical system. To simplify the analysis of complex systems by reducing the number of independent variables needed to describe their motion. To classify constraints (holonomic, non-holonomic, scleronomic, rheonomic). To express constraints as mathematical equations or inequalities. To eliminate redundant degrees of freedom in a system's description. To allow the ability to focus on the essential movements of a system. To provide a method for analyzing forces and equilibrium in a system without explicitly considering constraint forces. To develop a tool for deriving equations of motion based on the principle of virtual displacements. To define virtual displacements as infinitesimal, hypothetical displacements that are consistent with the system's constraints. To calculate the virtual work done by forces during these virtual displacements. To establish that the total virtual work done by all forces in a system in equilibrium is zero. To allow for the analysis of forces without needing to solve for the constraint forces. To extend the principle of virtual work to dynamic systems, allowing for the analysis of motion under the influence of forces. To provide a foundation for deriving Lagrange's equations, a powerful tool for describing the motion of constrained systems. To combine the concept of virtual work with Newton's second law to account for inertial forces. To state that the virtual work done by the impressed forces and the inertial forces in a system is zero. To use D'Alembert's principle to derive Lagrange's equations by introducing generalized coordinates and the Lagrangian function ($L = T - V$, where T is kinetic energy and V is potential energy). To create a method of creating equations of motion using scalar energy values, rather than vector forces. To establish a systematic procedure for deriving equations of motion for complex mechanical systems. To simplify the analysis of systems with constraints by using generalized coordinates and the Lagrangian. Identify the degrees of freedom of the system and choose appropriate generalized coordinates. Express the kinetic and potential energies of the system in terms of these generalized coordinates and their time derivatives. Form the Lagrangian function ($L = T - V$).

To learn about:

- Constraints
- The concept of virtual work
- D'Alembert's Principlederivation of Lagrangian equation of motion from it
- Procedure for formation of Lagrange's Equations

STRUCTURE:

2.1 CONSTRAINTS:

Often the motion of a particle or system of particles is restricted by one or more conditions. The limitations on the motion of a system are called constraints and the motion is said to be constrained motion.

2.1.1. Holonomic Constraints:

Constraints limit the motion of a system and the number of independent coordinates, needed to describe the motion, is reduced. For example, if a particle is allowed to move on the circumference of a circle, then only one coordinate $q_1 = \theta$ is sufficient to describe the motion, because the radius (a) of the circle remains the same. If **r** is the position vector of the particle at any angular coordinate θ relative to the centre of the circle, then

$$|r| = a \text{ or } r - a = 0 \qquad (1)$$

Equation expresses the constraint for a particle in circular motion. Similarly, in the case of a particle, moving on the surface of a sphere, the correct coordinates are spherical coordinates r, θ and ϕ only vary. Therefore $q_1 = \theta$ and $q_2 = \phi$ are the two independent coordinates for the problem because the constraint is that the radius of the sphere (a) is constant (i.e., $|r| = a$). Since in the circular motion of the particle, one independent coordinate θ is needed, the number of degrees of freedom of the system is 1. For the particle, constrained to move on the surface of the sphere, two independent coordinates specify its motion and hence the degrees of freedom are 2.

Suppose the constraints are present in the system of N particles. If the constraints are expressed in the form of equations of the form

$$f(r_1, r_2, \dots, t) \qquad (2)$$

then they are called holonomic constraints. Let there be m number of such equations to describe the constraints in the particle system. Now, we may use these equations to eliminate m of the 3N coordinates and we only n independent coordinates to describe the motion, given by

$$n = 3N - m$$

The system is said to have n or $3N - m$ degrees of freedom. The elimination of the dependent coordinates can be expressed by introducing n = $3N - m$ independent variables $q_1, q_2, q_3, \dots, q_n$. These are referred as generalized coordinates.

Superfluous Coordinates: The idea of degrees of freedom makes it clear that when we are using, say rectangular cartesian coordinates, we have several redundant or superfluous coordinates, if there are holonomic constraints. This redundance and non-independence of the coordinates makes the problem complicated and this difficulty is resolved by using the generalized coordinates. For example, let us consider a body be thrown vertically upward with an initial velocity v_o. The body will move in a straight line. In cartesian coordinates, the motion will be represented as

$$x = 0, y = v_0 t - \frac{1}{2} g t^2, z = 0$$

Where X and Z axes are horizontal and Y-axis is in vertical direction. At different values of the time t, only y coordinate varies and x and z coordinates remain the same. Therefore, x and z coordinates are superfluous coordinates. In conclusion, we need only one coordinate q = y to describe the vertical motion.

2.1.2 Nonholonomic Constraints:

The constraints which are not expressible in the form of eq. (50) are called non-holonomic. For example, the motion of a particle, placed on the surface of a sphere of radius a, will be described by

$$|r| \geq a \text{ or } r - a \geq 0$$

In a gravitational field, where **r** is the position vector of the particle relative to the centre of the sphere. The particle will first slide down the surface and then fall off. The gas molecules in a container are constrained to move inside it and the related constraint is another example of nonholonomic constraints. If the gas container is in spherical shape with radius a and **r** is the position vector of a molecule. then the condition of constraint for the motion of molecules can be expressed as

$$|r| \leq a \text{ or } r - a \leq 0$$

It is to be mentioned that in holonomic constraints, each coordinate can vary independently of other. In a nonholonomic system, all the coordinates cannot vary independently and hence the number of degrees of freedom of the system is less than the minimum number of coordinates needed to specify the configuration of the system. We shall in general consider the holonomic systems.

Constraints are further described as (i) rheonomous and (ii) scleronomous. In the former, he equation of constraint contain the time as an explicit variable, while in the later they are not explicitly dependent on time. Constraints may also be classified as (i) conservative and (ii) dissipative. In case of conserve constraints, total mechanical energy of the system is conserved during the constrained motion and constraint forces do not do any work. In dissipative constraints, the constraint forces do work and the mechanical energy is not conserved. Time-dependent or rheonomic constraints are generally dissipative.

Forced Constraints:

Constraints are always related to forces which restrict the motion of the system. These forces are called forces of constraint. For example, the reaction force on a sliding particle on the surface of a sphere is the force of constraint. In case of a rigid body, the inertial forces of action and reaction between any two particles are the forces of constraint. -Constraint force in a simple pendulum is the tension in the string. When a bead slides on a wire, the reaction force exerted by the wire on the bead is the force of constraint. These forces of -constraint are elastic in nature and usually appear at the surface of contact because the motion due to external applied forces' is hindered by the contact. However, Newton has not given any prescription to calculate these forces of constraint.

Usually, the .constraint forces act in a direction perpendicular to the surface of constraints while 'the motion of the object is parallel to the surface. In such cases, the work done by the forces of constraint is these constraints are termed as *workless* and may be called as *ideal constraints.* For example, when a particle slides on a frictionless horizontal surface, the force of constraint is normal to the surface. There are examples, where the constraint force does work. When a body slides on a frictional surface, the work is done by the force of constraint (frictional force) for real displacements.

By definition, the external or applied\forces are all known forces. In the solution of dynamical problems either we have to determine all the forces of-constraints or we should eliminate them from final equations. If we want to use Newton's form of equations, the forces of constraints are to be determined. We discuss below the difficulties, introduced by such an approach and how to remove them.

Difficulties introduced by the Constraints and their Removal:

Two types of difficulties are introduced by constraints in the solution of mechanical problems:

(1) Let us consider a system of *N* interacting particles. The force on the ith particle is given by

$$F_i = F_i^e + \sum_{j=1}^{N} F_{ij}$$

where F_i^e stands for an external force and $F_{ij}$is the internal (constraint) force on the i[th] particle due to j[th] particle. The equation of motion of the i[th] particle, in view of Newton's second law, is

$$m_i \frac{d^2 r_i}{dt^2} = F_i^e + \sum_{j=1}^{N} F_{ij} \tag{3}$$

where i= 1, 2,...., N. Thus eq. (3) represents a set of N equations. The coordinates $r_i$are connected by equations of constraints of the form:

$$f(r_1, r_2, \dots, r_n, t) = 0$$

Hence the coordinates $r_1, r_2, \dots, r_N$ of various particles are no longer all independent. This means that N equations represented by (3) are not all independent and therefore, the equations of motion are to be written again taking into consideration the equations of the constraints.

(2) The second difficulty introduced by the constraints is that several times the constraint forces are not known initially and they are among the unknowns of the problem. In fact, these unknown constraint forces are to be obtained from the solution of the problem which we are seeking and thus introduce complications in obtaining the solution. For example, if a bead is moving on a wire, the force (of constraint) which the wire exerts on the bead is not known in the beginning of the problem.

In case of holonomic constraints, as discussed already, the first difficulty is solved by introducing $n = 3N - m$ generalized coordinates, where m is the number of equations of constraints in N particle system. In order to remove the second difficulty, namely the forces of constraint are not known initially, we formulate the mechanics in such a way that the forces of constraint disappear. We require then only the known applied forces. Such an approach is due to D'Alembert which uses the ideas of virtual displacement and virtual work.

2.2 PRINCIPLE OF VIRTUAL WORK:

In order to investigate the properties of a system, we can imagine arbitrary instantaneous change in the position vectors of the particles of the system *e.g.,* virtual displacements. An infinitesimal virtual displacement of i[th] -particle of a system of N particles is denoted by δr_i. This is the displacement of position coordinates only and does not involve a *variation of time i.e.,*

$$\delta r_i = \delta r_i(q_1, q_2, \dots, q_n) \tag{4}$$

Suppose the system is in equilibrium, then the total force on any particle is zero *i.e.,*

$$F_i = 0, \qquad i=1, 2, \dots, N$$

The virtual work of force F_i in the virtual displacement δr_i will also be zero *i.e.,*

$$\delta W_i = F_i \cdot \delta r_i = 0$$

Similarly, the sum of virtual work for all the particles must vanish *i.e.*

$$\delta W = \sum_{i=1}^{N} F_i \cdot \delta r_i = 0 \qquad (5)$$

This result represents the ***principle of virtual work*** which states that ***the work done is zero in the case of an arbitrary virtual displacement of a system from a position of equilibrium***

The total force F_i on the ith particle can be expressed as $F_i = F_i^a + \mathrm{f}_i$

where F_i^a is the applied force and f_i is the force of constraint

Hence eq. (5) assumes the form

$$\sum_{i=1}^{N} F_i^a \cdot \delta r_i + \sum_{i=1}^{N} \mathrm{f}_i \cdot \delta r_i = 0$$

We restrict ourselves to the systems where the virtual work of the forces of constraints is zero, *e.g.,* in case of .a rigid body. Then

$$\sum_{i=1}^{N} F_i^a \cdot \delta r_i = 0 \qquad (6)$$

i.e., for equilibrium of a system, the virtual work of applied forces is zero. We see that the principle of virtual work deals with the statics of a system of particles. However, we want a principle to deal with the general motion of the system and such a principle was developed by D'Alembert.

2.3 D'ALEMBERT'S PRINCIPLE:

According to Newton's second law of motion, the force acting on the ith particle is given by

$$F_i = \frac{dp_i}{dt} = \dot{p}_i$$

This can be written as

$$F_i - \dot{p}_i = 0, \qquad i=1, 2, \ldots, N$$

These equations mean that any particle in the system is in equilibrium under a force, which is equal to the actual force F_i plus a reversed effective force $\dot{p}_i$. Therefore, for virtual displacements δr_i.

$$\sum_{i=1}^{N} (F_i - \dot{p}_i) \cdot \delta r_i = 0$$

But $F_i = F_i^a + \mathrm{f}_i$, then

$$\sum_{i=1}^{N} (F_i^a - \dot{p}_i) \cdot \delta r_i + \sum_{i=1}^{N} \mathrm{f}_i \cdot \delta r_i = 0$$

Again, we restrict ourselves to the systems for which the virtual work of the constraints is zero, *i.e.*, $\sum_{i=1}^{N} \mathrm{f}_i \cdot \delta r_i = 0$. Then

$$\sum_{i=1}^{N}(F_i^a - \dot{p}_i) \cdot \delta r_i = 0 \tag{7}$$

This is known as ***D'Alembert's principle.*** Since the forces of constraints do not appear in the equation and hence now, we can drop the superscript. Therefore, the D'Alembert's principle may be written as

$$\sum_{i=1}^{N}(F_i - \dot{p}_i) \cdot \delta r_i = 0 \tag{8}$$

2.4 LAGRANGE'S EQUATIONS FROM D'ALEMBERT'S PRINCIPLE:

Consider a system of *N* particles. The transformation equations for the position vectors of the particle

$$r_i = r_i\,(q_1, q_2, \ldots, q_k, \ldots, q_n, t) \tag{9}$$

where *t* is the time and q_k (k = 1, 2,...,n) are the generalized coordinates.

Differentiating eq. (9) with respect to *t*, we obtain the velocity of the i[th] particle, *i.e.*,

$$\frac{dr_i}{dt} = \frac{\partial r_i}{\partial q_1}\frac{dq_1}{dt} + \frac{\partial r_i}{\partial q_2}\frac{dq_2}{dt} + \cdots + \frac{\partial r_i}{\partial q_k}\frac{dq_k}{dt} + \cdots + \frac{\partial r_i}{\partial q_n}\frac{dq_n}{dt} + \frac{\partial r_i}{\partial t}$$

Or

$$v_i = \dot{r}_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t} \tag{10}$$

where $\dot{q}_k$ are the generalized velocities.

The virtual displacement is given by

$$\delta r_i = \frac{\partial r_i}{\partial q_1}\,dq_1 + \frac{\partial r_i}{\partial q_2}\,dq_2 + \cdots + \frac{\partial r_i}{\partial q_k}\,dq_k + \cdots + \frac{\partial r_i}{\partial q_n}\,dq_n$$

$$\delta r_i = \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\,\delta q_k \tag{11}$$

Since by definition the virtual displacements do not depend: on time.

According to D'Alembert's principle,

$$\sum_{i=1}^{N}(F_i - \dot{p}_i) \cdot \delta r_i = 0 \tag{12}$$

Here $\sum_{i=1}^{N} F_i \cdot \delta r_i = \sum_{i=1}^{N} F_i \cdot \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\,\delta q_k = \sum_{i=1}^{N}\sum_{k=1}^{n}\left[F_i \cdot \frac{\partial r_i}{\partial q_k}\right]\,\delta q_k$

$$= \sum_{k=1}^{n} G_k\,\delta q_k \tag{13}$$

Where $G_k = \sum_{i=1}^{N} F_i \cdot \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N} \left[F_{x_i} \frac{\partial x_i}{\partial q_k} + F_{y_i} \frac{\partial y_i}{\partial q_k} + F_{z_i} \frac{\partial z_i}{\partial q_k}\right]$ (14)

These are called the components of **generalized force** associated with the generalized coordinates q_k. This may be mentioned that as the dimensions of the generalized coordinates need not be those of length, similarly the generalized force components $G_k$may have dimensions different than those of force. However, the dimensions of G_k, δq_k are those of work. For example, if δq_khas the dimensions of length, G_k will have the dimensions of force; if δq_k has the dimensions of angle *(θ),* G_k will have the dimensions of torque (τ).

Further

$$\sum_{i=1}^{N} \dot{p}_i \cdot \delta r_i = \sum_{i=1}^{N} m_i \ddot{r}_i \cdot \sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k} \delta q_k = \sum_{k=1}^{n} \left[\sum_{i=1}^{N} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_k}\right] \delta q_k \quad (15)$$

Now $$\sum_{i=1}^{N} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N} [\frac{d}{dt}(m_i \dot{r}_i \cdot \frac{\partial r_i}{\partial q_k}) - m_i \dot{r}_i \cdot \frac{d}{dt} (\frac{\partial r_i}{\partial q_k})] \quad (16)$$

It is easy to prove that

$$\frac{d}{dt}\left(\frac{\partial r_i}{\partial q_k}\right) = \frac{\partial}{\partial q_k}\left(\frac{d r_i}{dt}\right) = \frac{\partial \mathrm{v}_i}{\partial q_k} \quad (16a)^*$$

And$\frac{\partial r_i}{\partial q_k} = \frac{\partial \mathrm{v}_i}{\partial \dot{q}_k}$ (16b)*

Therefore, eq (16) is

$$\sum_{i=1}^{N} m_i \ddot{r}_i \cdot \frac{\partial r_i}{\partial q_k} = \sum_{i=1}^{N} \left[\frac{d}{dt}\left[m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_k}\right] - m_i v_i \cdot \frac{\partial v_i}{\partial q_k}\right] \quad (17)$$

Substituting eq (15), we get

$$\sum_{i=1}^{N} \dot{p}_i \cdot \delta r_i = \sum_{k=1}^{n} \sum_{i=1}^{N} \left[\frac{d}{dt}\left[m_i v_i \cdot \frac{\partial v_i}{\partial \dot{q}_k}\right] - m_i v_i \cdot \frac{\partial v_i}{\partial q_k}\right] \delta q_k$$

$$= \sum_{k=1}^{n} \left[\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_k}\left(\sum_{i=1}^{N} \frac{1}{2} m_i (v_i \cdot v_i)\right)\right\} - \frac{\partial}{\partial q_k}\left\{\sum_{i=1}^{N} \frac{1}{2} m_i (v_i \cdot v_i)\right\}\right] \delta q_k$$

$$= \sum_{k=1}^{n} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right] \delta q_k \quad (18)$$

where $\sum_{i=1}^{N} \frac{1}{2} m_i (v_i \cdot v_i) = T$ is the **kinetic energy** of the system

Substituting for $\sum_i F_i \cdot \delta r_i$ from (12) and $\sum_i \dot{p}_i \cdot \delta r_i$ from (18) in eq.(11), the D'Alembert's principle becomes

$$\sum_{k=1}^{n} \left[\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}\right\} - G_k\right] \delta q_k = 0 \quad (19)$$

As the constraints are holonomic, it means that any virtual displacement δq_k is independent of δq_jTherefore, the coefficient in the square bracket for each δq_kmust be zero, *i.e.*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) - \frac{\partial T}{\partial q_k} - G_k = 0 \; or \; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) - \frac{\partial T}{\partial q_k} = G_k \tag{20}$$

This represents the *general form of Lagrange's equations.*

For a **conservative system,** the force is derivable from a scalar potential V

$$F_i = \nabla_i V = -\hat{\imath}\frac{\partial V}{\partial x_i} - \hat{\jmath}\frac{\partial V}{\partial y_i} - \hat{k}\frac{\partial V}{\partial z_i} \tag{21}$$

Hence from eq. (14), the generalized force components are

$$G_k = -\sum_{i=1}^{N}\left[\frac{\partial V}{\partial x_i}\frac{\partial x_i}{\partial q_k} + \frac{\partial V}{\partial y_i}\frac{\partial y_i}{\partial q_k} + \frac{\partial V}{\partial z_i}\frac{\partial z_i}{\partial q_k}\right] \tag{22}$$

Clearly the right hand side of the above equation is the partial derivative of — V with respect to q_k *i.e.,*

$$G_k = -\frac{\partial V}{\partial q_k} \tag{23}$$

Thus eq. (20) assumes the form

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) - \frac{\partial T}{\partial q_k} = -\frac{\partial V}{\partial q_k} \tag{24}$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q_k}}\right) - \frac{\partial (T-V)}{\partial q_k} = 0 \tag{25}$$

Since the scalar potential V is the function of generalized coordinates q_k*only* not depending on generalized velocities, we can write eq. (25) as

$$\frac{d}{dt}\left[\frac{\partial (T-V)}{\partial \dot{q_k}}\right] - \frac{\partial (T-V)}{\partial q_k} = 0 \tag{26}$$

We define a new function given by

$$L = T - V \tag{27}$$

which is called the *Lagrangian* of the system.; Thus, eq. (26) takes the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{28}$$

for $k = 1, 2, \ldots, n$

These equations are known as **Lagrange's equations** for conservative system. They are n in number and there is one equation for each generalized coordinate. In order to solve these equations, we must know the Lagrangian function $L = T - V$ in the appropriate generalized coordinates.

2.4.1 PROCEDURE FOR FORMATION OF LAGRANGE'S EQUATIONS:

The Lagrangian function L of a system is given by

$$L = T - V \tag{29}$$

In order to form L, kinetic energy T and potential energy V are to be written in generalized coordinates. This is then substituted in the Lagrangian equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad (30)$$

to obtain the equations of motion of the system. This involves first to find the partial derivatives of L, *i.e:*, $\frac{\partial L}{\partial q_k}$and $\frac{\partial L}{\partial \dot{q}_k}$and then to put their values in eq. (30)..

Kinetic Energy in Generalized Coordinates: The transformation equations are used to transform T from cartesian coordinates to generalized coordinates. Therefore

$$T = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i \dot{r}_i{}^2 = \sum_i \frac{1}{2}\left(\sum_{k=1}^{n} \frac{\partial r_i}{\partial q_k}\dot{q}_k + \frac{\partial r_i}{\partial t}\right)^2$$

or $$T = M_0 + \sum_k M_k \dot{q}_k + \frac{1}{2}\sum_{kl} M_{kl} q_k \dot{q}_l \qquad (31)$$

Where $$M_0 = \sum_k \frac{1}{2} m_i \left(\frac{\partial r_i}{\partial t}\right)^2, \quad M_k = \sum_i m_i \frac{\partial r_i}{\partial t} \cdot \frac{\partial r_i}{\partial q_k}$$

and $$M_{kl} = \sum_i m_i \frac{\partial r_i}{\partial q_k} \cdot \frac{\partial r_i}{\partial q_l}$$

Thus, we see from (31) that in the expression for kinetic energy, first term is independent of generalized velocities, while second and third terms are linear and quadratic in generalized velocities respectively. For scleronomic systems, the transformation equations do not contain time explicitly. So that

$$v_i = \dot{r}_i = \sum_k \frac{\partial r_i}{\partial q_k}\dot{q}_k$$

Therefore $$T = \sum_i \frac{1}{2} m_i v_i^2 = \frac{1}{2}\sum_{kl} M_{kl}\, \dot{q}_k \dot{q}_l \qquad (32)$$

In such a case, the expression for T is a homogeneous quadratic form in generalized velocities.

2.5 SUMMARY:

The relations, which restrict the motion of particles, are called constraints. The constraints are divided into holonomous and non-holonomous types. Non-holonomous constraints are further divided into' scleronomous and rheonomous constraints. Constraints are associated with force called constraint forces. However, the laws of mechanics are so formulated so that the work done by the, forces of constraints are zero.

Constraints:

- These are limitations on a system's motion, simplifying analysis by reducing the number of needed variables. They're expressed mathematically and categorized (holonomic, non-holonomic, etc.).

Virtual Work:

- This concept involves hypothetical, infinitesimal displacements (virtual displacements) to analyze forces without dealing directly with constraint forces. The principle states that the total virtual work in equilibrium is zero.

D'Alembert's Principle:

- This extends virtual work to dynamic systems by including inertial forces. It states that the virtual work of impressed and inertial forces is zero. This principle is the foundation for deriving Lagrange's equations.

Lagrange's Equations:

- These equations, derived from D'Alembert's principle, describe a system's motion using generalized coordinates and the Lagrangian (kinetic minus potential energy). They simplify complex systems, especially those with constraints.

Procedure for Formation of Lagrange's Equations:

1) Identify degrees of freedom and generalized coordinates.
2) Express kinetic and potential energy in terms of these coordinates.
3) Form the Lagrangian (L = T - V).
4) Apply Lagrange's equations: dtd(∂q˙i∂L)−∂qi∂L=0.
5) Solve the resulting equations of motion.

This procedure produces equations of motion using scalar energy values.

2.6 TECHNICAL TERMS:

"D'Alembert's Principle, Lagrangian equations. virtual work.

2.7 SELF-ASSESSMENT QUESTIONS:

1) What are constraints? Give specific examples to explain the forces of constraints.
2) Derive Lagrange equations of motion.
3) Derive Lagrange's equation of motion from D'Alembert's principle.

2.8 SUGGESTED READINGS:

1) Classical Mechanics: H. Goldstein
2) Mechanics: Simon
3) Mechanics: Gupta, Kumar and Sharma.

Prof. R.V.S.S.N. Ravi Kumar

LESSON 3

APPLICATIONS OF LAGRANGE EQUATION

3.0 AIM AND OBJECTIVES:

To demonstrate the versatility and power of Lagrange's equations in solving a wide range of problems in classical mechanics. To simplify the analysis of complex systems with constraints, where traditional Newtonian methods become cumbersome. To provide a method of analyzing systems using scalar energy values, rather than vector forces. To apply Lagrange's equations to analyze the motion of various mechanical systems, including:

- Simple harmonic oscillators.
- Pendulums (simple, double, spherical).
- Rotating bodies.
- Systems with multiple degrees of freedom.
- Systems with complex constraints.
- To illustrate the advantages of using generalized coordinates and the Lagrangian formulation in simplifying problem-solving.
- To derive equations of motion for systems that are difficult to analyze using Newton's laws directly.
- To show the ability to apply the equations to systems that include non-conservative forces.
- To extend the Lagrangian formulation to describe the motion of charged particles in electromagnetic fields.
- To provide a framework for analyzing the interaction between charged particles and electromagnetic fields using the Lagrangian approach.
- To allow the inclusion of electromagnetic forces into the Lagrangian formulation.
- To construct the Lagrangian function for a charged particle in an electromagnetic field, including the contributions from the particle's kinetic energy and its interaction with the electromagnetic potentials (scalar and vector potentials).To derive the equations of motion for the charged particle using Lagrange's equations, which will include the Lorentz force.To demonstrate how the electromagnetic forces can be incorporated into the Lagrangian framework through the use of electromagnetic potentials.To show the link between classical mechanics and electromagnetism through the Lagrangian formalism.To provide a base for more advanced studies into relativistic electromagnetism.

To learn about:

- Applications of Lagrange equation
- Lagrangian for a Charged Particle Movingin an Electromagnetic Field

STRUCTURE:

3.1 APPLICATIONS OF LAGRANGE EQUATION:

3.1.1 LINEAR HARMONIC OSCILLATOR:

A Harmonic oscillator is a particle which is bound to an equilibrium position by a force which is proportional to the displacement from that position.

Thus we have,

$$\text{Force} = -\gamma x = -\frac{dU}{dx} \qquad (1)$$

where $\gamma$is the spring constant.

The potential is expressed as,

$$U(x) = \frac{1}{2}\gamma x^2 \qquad (2)$$

The linear harmonic oscillator can then be visualized on a mass connected to a spring of spring constant γon shown in Fig. 3.1.

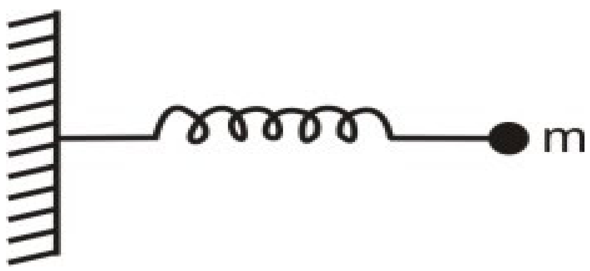

Fig. 3.1

The time-independent Schrödinger equation is given by,

$$\frac{\hbar^2}{2m}\frac{d^2\varphi}{dx^2}+\frac{1}{2}\gamma x^2\varphi = E\varphi$$

or,

$$\frac{d^2\varphi}{dx^2}+\frac{2m}{\hbar^2}\left(E-\frac{1}{2}\gamma x^2\right)\varphi = 0 \qquad (3)$$

To solve equation (3), we consider a dimension less quantity,

$$y = \left(\frac{m\gamma}{\hbar^2}\right)^{\frac{1}{4}}x \qquad (4)$$

And

$$\lambda = \frac{2}{\hbar}\left(\frac{m}{\gamma}\right)^{\frac{1}{2}}E \qquad (5)$$

using (5) and (4)

$$\frac{d^2\varphi}{dx^2}+(\lambda-y^2)\varphi = 0 \qquad (6)$$

For large values of y, we can neglect λ we get equation (6) on,

$$\frac{d^2\varphi}{dx^2}-y^2\varphi = 0 \qquad (7)$$

Equation (7) is satisfied approximately by the solution,

$$\varphi(y) = e^{-y^2/2} \qquad (8)$$

Substituting equation (8) in (7) we get,

$$\frac{d^2\varphi}{dy^2}+(1-y^2)\varphi = 0 \qquad (9)$$

This indicates that equation (8) satisfied equation (7) approximately and hence we consider the exact solution as,

$$\varphi(y) = e^{-y^2/2}\,\xi(y) \qquad (10)$$

Putting the value of $\varphi(y)$ from (10) in (6)

$$\frac{d^2\xi}{dy^2}-2y\,\frac{d\xi}{dy}+(\lambda-1)\xi = 0 \qquad (11)$$

The trick next is to linearize the above equation.

Equation (11) can be solved by using the power series method.

Let the trial solution be,

$$\xi(y) = \sum_{n=0}^{\infty} a_n y^n \qquad (12)$$

$$\frac{d^2\xi}{dy^2} = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+1}\,2y^n \qquad (13)$$

$$-2y\,\frac{d\xi}{dy} = \sum_{n=0}^{\infty} -2na_n\,y^n \qquad (14)$$

$$(\lambda-1)\xi = \sum_{n=0}^{\infty}(\lambda-1)a_n\,y^n \qquad (15)$$

∴ Putting equation (13), (14) and (15) in (11),

$$\sum_{n=0}^{\infty}[(n+1)(n+2)\,a_{n+2} - (\lambda-1-2n)\,a_n\,]y^n=0$$

This equation must hold for all values of ξ, and therefore the coefficient of each power of ξmust vanish separately.

This gives in the recursion relation between a_{n+2} and a_n ,

$$a_{n+2} = \frac{2n-\lambda+1}{(n+1)(n+2)}\,a_n \qquad (16)$$

It seems that knowing $a_0$and a_1, a_2, a_3, ... can be calculated by using equation (16),

$$a_2 = \frac{-(\lambda-1)}{2!}\,a_0 \qquad a_3 = \frac{-(\lambda-3)}{3!}\,a_1$$

$$a_4 = \frac{(\lambda-1)(\lambda-5)}{4!}\,a_0 \qquad a_5 = \frac{(\lambda-3)(\lambda-7)}{5!}\,a_1$$

Thus we can write equation (12) as,

$$\xi(y) = a_0\left[1-\frac{(\lambda-1)}{2!}y^2+\frac{(\lambda-1)(\lambda-5)}{4!}y^4+\cdots\right]+a_0\left[y-\frac{(\lambda-3)}{3!}+\frac{(\lambda-3)(\lambda-7)}{5!}+\cdots\right] \qquad (17)$$

If in the equation (16)$\lambda-1-2n$ should be zero for some value of the index n, then $a_{n+2} = 0$. But since a_{n+4} is a multiple of a_{n+2} so on, all the succeeding coefficients which are related to $a_n$by the recursion relation (16) would vanish, and one or the other bracketed series in equation (17) would terminate to become a polynomial of degree n.

This occurs, when,

$$\lambda-1-2n = 0$$

$$or, \lambda = 2n+1 \text{ (n = 0, 1, 2, ...)} \qquad (18)$$

Energy Quantization:

We have obtained the condition when the wave function is acceptable as

$$\lambda = 2n+1 \text{ (n = 0, 1, 2, ...)}$$

$$\lambda = \frac{2}{\hbar}\left[\frac{m}{\gamma}\right]^{1/2} E$$

again $\lambda$was

$$E_n = \frac{\hbar}{2}\omega(2n+1) \qquad \left[\text{As } \omega = \left[\frac{\gamma}{m}\right]^{1/2}\right]$$

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \qquad (19)$$

The variation of the energy levels is shown in the Fig. 3.2.

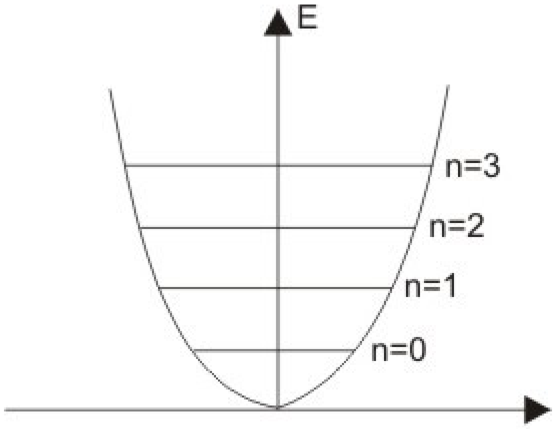

Fig. 3.2

3.1.2 Simple Pendulum:

The equation of motion for a simple pendulum of length l, operating in a gravitational field is¨ This equation can be obtained by applying Newton's Second Law (N2L) to the pendulum and then writing the equilibrium equation. It is instructive to work out this equation of motion also using Lagrangian mechanics to see how the procedure is applied and that the result obtained is the same. For this example, we are using the simplest of pendula, i.e. one with a massless, inertia less link and an inertia less pendulum bob at its end, as shown in Figure 3.3.

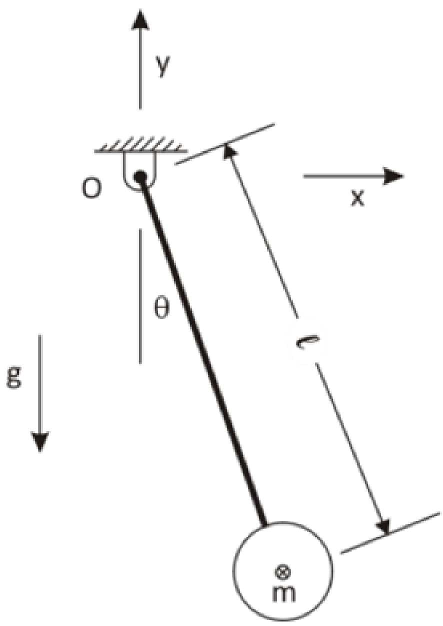

Fig. 3.3 Simple Pendulum

Lagrangian formulation The Lagrangian function is defined as

$$L = T - V$$

where T is the total kinetic energy and U is the total potential energy of a mechanical system.

To get the equations of motion, we use the Lagrangian formulation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = F_i$$

where q signifies generalized coordinates and F signifies non-conservative forces acting on the mechanical system. For the simplify pendulum, we assume no friction, so no non-conservative forces, so all Fi are 0. The aforementioned equation of motion is in terms of θ as a coordinate, not in terms of x and y. So we need to use kinematics to get our energy terms in terms of θ.

For T, we need the velocity of the mass.

$$v = l.\dot{\theta}$$

So

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(l.\dot{\theta}\right)^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

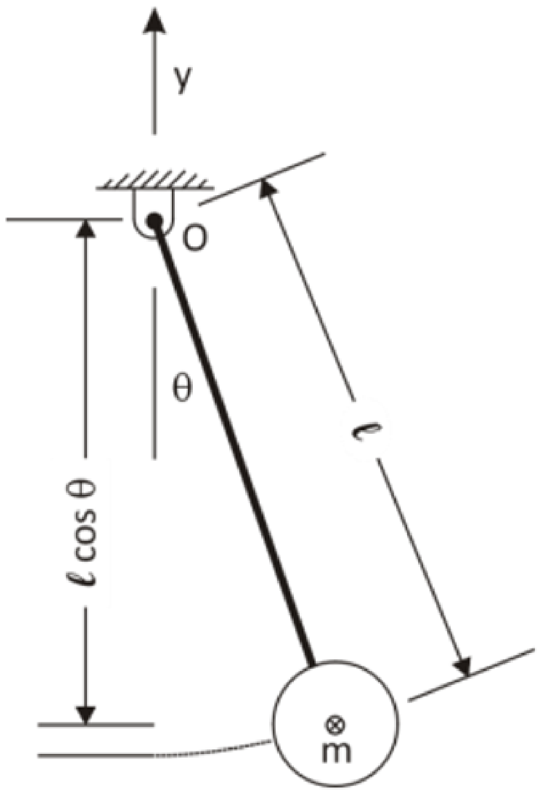

Fig. 3.4 Height for Potential Energy

The potential energy, U, depends only on the y-coordinate. Taking θ = 0 as the position where U = 0,

$$y = l - l\,.\,cos\theta = l\,.\,(1 - cos\theta)$$

Thus

$$U = m\,.\,g\,.\,y = m\,.\,g\,.\,l\,.\,(1 - cos\theta)$$

Now we have all the parts and pieces to complete the Lagrangian formulation. The Lagrangian function in terms of θ is

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - m\,.\,g\,.\,l\,.\,(1 - cos\theta)$$

The only generalized coordinate is q_i = θ. So

$$\frac{\partial L}{\partial \dot{\theta}} = m\,.\,l^2.\,\dot{\theta}$$

Continuing

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\,.\,l^2.\,\ddot{\theta}$$

Then $\frac{\partial L}{\partial \theta} = -m\,.\,g\,.\,l\,.\,sin\theta$

Now, putting these last two equations together

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = m\,.\,l^2.\,\ddot{\theta} + m\,.\,g\,.\,l\,.\,sin\theta = 0$$

Simplifying,

$$\ddot{\theta} + \frac{g}{l}\,.\,sin\,\theta = 0$$

This equation shows that under the condition that the angular amplitude is very small the motion of the pendulum is simple harmonic of time period.

$$T=2\pi\sqrt{\frac{l}{g}}$$

3.1.3 Compound Pendulum:

Any rigid body capable of oscillating in a vertical plane about a horizontal axis passing through any point (excepting the centre of gravity) of the bodyis called a compound pendulum.

Let the vertical plane of oscillation of the compound pendulum be the XY plane.

Let us choose the origin of the coordinate system as the point O through which the horizontal axis (the Xaxis) passes.

Let *G* be the position of the centre of gravity of the body when at rest.

OG= *l* (say)

On displacing the pendulum slightly from the position of rest and releasing, the pendulum begins to oscillate about the horizontal axis through O.

At any instant of time *t*, let *G'* be the new position of the centre of gravity and $G\hat{O}G'$ be equal, as shownin Figure5.

Thekinetic energyof thependulum atthe instant*t is*

$$T = \frac{1}{2}I(\dot{\theta})^2 \tag{20}$$

Where *I* is the moment of inertia of the pendulum about the axis of oscillation.

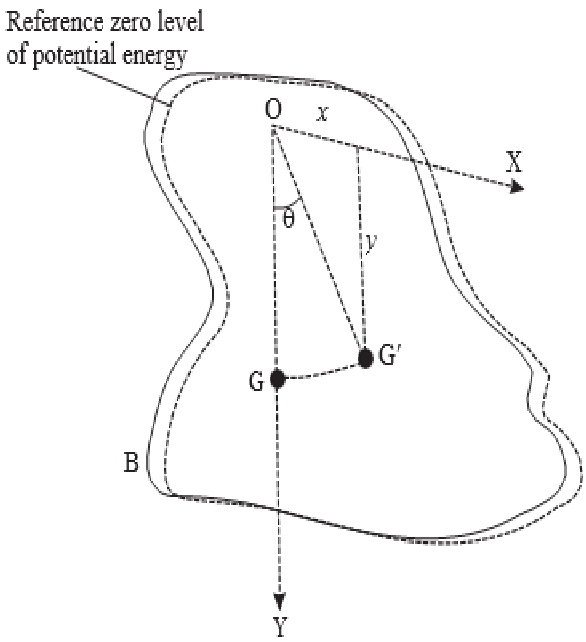

Fig. 3.5 Positions of Centre of Gravity

Taking the horizontal axis OX as the reference zero of potential energy, we get the potential energy of the pendulum at the instant*t*as

$$V = -\,mgy = -mgl\cos\theta \tag{21}$$

The Lagrangian of the pendulum is thus

$$L = T - V = \frac{1}{2}I(\dot{\theta})^2 + mglcos\theta \tag{22}$$

From Equation (22) we find that the only generalized coordinate for the pendulum is θ.We thus have the Lagrange's equation for the compound pendulum

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \tag{23}$$

From equation (22) the above equation gives

$$\frac{d}{dt}\left(\frac{1}{2}.I.2.\dot{\theta}\right) = -mgl\,sin\theta$$

$$I\ddot{\theta} + mgl\,sin\theta = 0$$

$$\ddot{\theta} + \frac{mgl\ sin\theta}{I} = 0 \qquad (24)$$

Considering$\dot{\theta}$smallEquation(24) reducesto

$$\ddot{\theta} = -\frac{mgl\ sin\theta}{I} \qquad (25)$$

Clearly,themotionofthependulumissimpleharmonictotimeperiod

$$T = 2\pi\sqrt{\frac{l}{mgl}} \qquad (26)$$

3.1.4. L-C Circuit:

The LC-circuit (or resonant circuit) is an electrical circuit that consists of an inductor, L, connected in series with a capacitor, C, see Fig. 1. The LC- circuit is an idealised model of the RLC- circuit where the resistance is assumed to be zero, thus no energy dissipation. A capacitor contains two conducting plates separated by a dielectric media such as glass or vacuum. By applying voltage over the plates, electric charge induce an electrical field between the plates, resulting in one plate receiving an excess of electrons while the other has too few. Therefore the two plates ends up having opposite charge.

At the point when the voltage source is removed, the capacitor keeps up its charge. The inductor, in its simplest form, is just a coils of wire. An inductor is a device that temporarily stores energy in the form of a magnetic field. The magnetic field is generated by the current flowing in the inductor. The inductor resists change in the current passing through. Assuming that the capacitor is charged and connected in series with an inductor, since there is no voltage source the capacitor will start losing its charge resulting in a current flow through the circuit. The inductor is acting as a resistance to the current change, meaning a slower rate at which the capacitor discharge.

At some point in time the capacitor will be completely discharged, all of the charge is moving as current in the circuit. The conductor counteracts the change of current by inducing its own current, forcing the charges to charge the capacitor. And the cycle begins again, only this time the current flows in the opposite direction. Solving the dynamics of the system by using the Kirchhoff's Voltage low, assuming conversation of the total energy, we get the following equations.

$$\Delta V_C + \Delta V_L = 0 \qquad (27)$$

$$\frac{Q}{C} + L\frac{dI}{dt} = 0 \qquad (28)$$

$$\frac{Q}{C} + L\frac{d^2I}{dt^2} = 0 \qquad (29)$$

Solving for Q

$$Q = Q_0 Cos(\omega_0 t) \qquad (30)$$

Where Q_0 is the charge stored in the capacitor at time t=0 and $\omega_0 = 1/\sqrt{LC}$ is the resonant frequency of the system.

Note that the capacitor acts like a source potential energy given by U = Q^2/LC and we can interpret T =$L\frac{\dot{Q}^2}{2}$ as a kinetic energy term. We can therefore attempt at writing a Lagrangian describing the system

$$\mathcal{L} = T - U = \frac{L\dot{Q}^2}{2} - \frac{Q^2}{2C}. \tag{31}$$

The Euler -Lagrange equations given in terms of the generalized coordinate Q and $\dot{Q}$are then

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{Q}}\right) - \frac{\partial \mathcal{L}}{\partial Q} = 0, \tag{32}$$

Where Q is the charge and $\dot{Q}$is the current. Inserting the Lagrangian L gives the following equation of motion for the charge

$$L\ddot{Q} + \frac{Q}{C} = 0, \tag{33}$$

This is same as (29), therefore we get again

$$Q = Q_0 \cos(\omega_0 t). \tag{34}$$

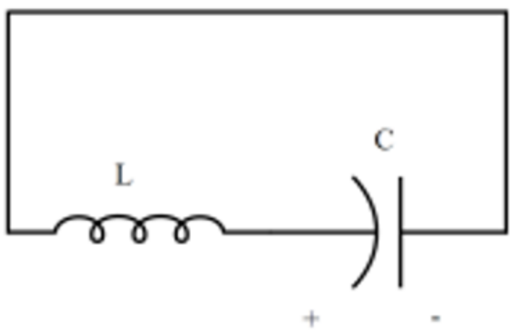

Fig. 3.6: Schematic of the LC-circuit

3.2. GENERALIZED POTENTIAL (VELOCITY DEPENDENT POTENTIAL) - LAGRANGIAN FOR A CHARGED PARTICLE MOVING-IN AN ELECTROMAGNETIC FIELD

In general, the Lagrange's equations can be written as

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = G_k \tag{35}$$

For a conservative system, $G_k = -\frac{\partial V}{\partial q_k}$ and then the Lagrange's equations in the usual form are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} = 0 \textit{ and } L = T - V \tag{36}$$

However, Lagrange's equations can be put in the form (94), provided the generalized forces are obtained from a function $U(q_k, \dot{q_k})$given by

$$G_k = -\frac{\partial U}{\partial q_k} + \frac{d}{dt}\left(\frac{\partial U}{\partial \dot{q_k}}\right) \tag{37}$$

In such a case, $L = T — U$

(38)

where $U(q_k, \dot{q_k})$is called ***velocity dependent potential*** or ***generalized potential.*** This type of case occurs in case of a charge moving in an electromagnetic field.

In S.I. system, two of the Maxwell's field equations are

div B = 0 and curl E + $\frac{\partial B}{\partial t}$=0

$$\text{or } \nabla \cdot B = 0 \text{ and } \nabla \times \mathbf{E} + \frac{\partial B}{\partial t} = 0 \tag{39}$$

where E and B are electric field and magnetic field vectors respectively

The force acting on a charge q, moving with velocity v in an electric field E and magnetic induction **B** is given by

$$F = q\,(E + \mathrm{v} \times B) \tag{40}$$

Since $\nabla \cdot B = 0$in eq. (39) and hence **B** can be expressed as curl of a vector *i.e.*

$$B = \nabla \times A \tag{41}$$

$$\nabla \times E + \frac{\partial}{\partial t}\nabla \times A = 0 \; or \; \nabla \times \left(E + \frac{\partial A}{\partial t}\right) = 0 \tag{42}$$

Hence, we can express the vector quantity $\left(E + \frac{\partial A}{\partial t}\right)$ as the gradient of a scalar function ϕ

$$E + \frac{\partial A}{\partial t} = -\nabla\phi \; or \; E = -\nabla\phi - \frac{\partial A}{\partial t} \tag{43}$$

Substituting for E from (43) in (40), we obtain

$$F = q(-\nabla\phi - \frac{\partial A}{\partial t} + \mathrm{v} \times \nabla \times A) \tag{44}$$

The terms in eq. (44) can be written in a more convenient form

Let us consider the x-component. Since $\nabla\phi = \hat{\imath}\frac{\partial \phi}{\partial x} - \hat{\jmath}\frac{\partial \phi}{\partial y} - \hat{k}\frac{\partial \phi}{\partial z}$ x-component of $\nabla\phi$ is $\frac{\partial \phi}{\partial x}$. Also,

$$(\mathrm{v} \times \nabla \times A)_x = v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

We add and subtract the term $v_x \frac{\partial A_x}{\partial x}$. Then

$$(\mathrm{v} \times \nabla \times A)_x = v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial x} + v_z \frac{\partial A_z}{\partial x} - v_x \frac{\partial A_x}{\partial x} - v_y \frac{\partial A_x}{\partial y} - v_z \frac{\partial A_x}{\partial z} \quad (45)$$

However, $\frac{dA_x}{dt} = \frac{\partial A_x}{\partial x}\frac{dx}{dt} + \frac{\partial A_y}{\partial y}\frac{dy}{dt} + \frac{\partial A_z}{\partial z}\frac{dz}{dt} + \frac{\partial A_x}{\partial t} = v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t}$

Where $$v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_x}{\partial y} + v_z \frac{\partial A_x}{\partial z} = \frac{dA_x}{dt} - \frac{\partial A_x}{\partial t} \quad (46)$$

Further $$\frac{\partial}{\partial x}(v \cdot A) = \frac{\partial}{\partial x}\left(v_x A_x + v_y A_y + v_z A_z\right)$$

$$= v_x \frac{\partial A_x}{\partial x} + v_y \frac{\partial A_y}{\partial y} + v_z \frac{\partial A_z}{\partial z} \quad (47)$$

Substituting from (46) and (47) in (45), we get

$$(\mathrm{v} \times \nabla \times A)_x = \frac{\partial}{\partial x}(v \cdot A) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t} \quad (48)$$

Hence,we know the x-component of the force **F** is

$$F_x = q\left(-\frac{\partial \phi}{\partial x} - \frac{\partial A_x}{\partial t} + \frac{\partial}{\partial x}(v \cdot A) - \frac{dA_x}{dt} + \frac{\partial A_x}{\partial t}\right) = q\left(-\frac{\partial}{\partial x}(\phi - v \cdot A) - \frac{dA_x}{dt}\right) \quad (49)$$

Since $\frac{\partial}{\partial v_x}(v \cdot A) = \frac{\partial}{\partial v_x}\left(v_x A_x + v_y A_y + v_z A_z\right) = A_x$ and scalar potential ϕ *is* independent of v_x , we have

$$-\frac{dA_x}{dt} = \frac{d}{dv_x}(\phi - v \cdot A)$$

Therefore, $F_x = q\left[-\frac{\partial}{\partial x}(\phi - v \cdot A) + \frac{d}{dt}\left\{\frac{\partial}{\partial v_x}(\phi - v \cdot A)\right\}\right]$ (50)

We define a ***generalized potential U,*** given by

$$U = q(\phi - v \cdot A) \quad (51)$$

which is a ***velocity dependent potential***. Therefore, eq. (50) takes the form

$$F_x = -\frac{\partial U}{\partial x} + \frac{d}{dt}\frac{\partial U}{\partial v_x} \quad (52)$$

The Lagrarage's equations (35) in this case take the form

$(q_k = x, \dot{q_k} = \dot{x} = v_x \text{and} G_k = F_x)$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial v_x}\right) - \frac{\partial T}{\partial x} = F_x \tag{53}$$

Substituting F_x from (53) in (52), we get the Lagrange's equation as

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}}(T-U)\right) - \frac{\partial}{\partial x}(T-U) = 0$$

or

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \tag{54}$$

Where $L = T - U = T - q\,\phi + qv.A$ (55)

Eq. (55) gives the Lagrangian for a charged particle moving in an electromagnetic field.

Note: In Gaussian C.G.S. system q is to be replaced by q/c in eqs. (39) and (40), where *c* is the speed of light. Therefore, the expression for generalized potential is obtained to be $U = q\,\phi + \frac{q}{c}(v \bullet \mathrm{A})$

3.3 SUMMARY:

In this unit we have recollected various applications of Applications of Lagrange equation and also learned about dissipation functions and application Lagranges equation for velocity dependent potentials as an example of non-conservative system.

3.4 TECHNICAL TERMS:

Linear Harmonic Oscillator, Simple Pendulum, Compound Pendulum, L-C Circuit.

3.5 SELF-ASSESSMENT QUESTIONS:

1) What are Lagrangian applications?
2) Discuss about velocity-dependent potential.

3.6 SUGGESTED READINGS:

1) Classical Mechanics by H.Goldstein.
2) Fundamentals of Classical Mechanics by J.C. Upadhyaya.
3) Classical Mechanics by G. Aruldhas, PHI Publishers.
4) The Theory of relativity and applications, Allen Rea.

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LESSON-4

HAMILTON’S MECHANICS

4.0 AIM AND OBJECTIVES:

To learn about

- Deduction of Hamilton’s principle from D’Alembert’s principle
- modified Hamilton’s principle
- Hamilton’s principle and Lagrange’s equations

To reformulate Newton's second law in a way that's more convenient for dealing with constrained systems. To provide a foundation for deriving equations of motion in generalized coordinates.To express the dynamics of a system in terms of "virtual displacements," which are infinitesimal changes in the system's configuration. To effectively handle constraint forces by incorporating them into the equations of motion. To provide a stepping stone towards the development of Lagrangian mechanics. To provide a variational principle that describes the evolution of a dynamical system.To express the laws of motion in terms of an integral quantity called the "action." To establish that the actual path taken by a system between two points in time is the one that minimizes (or makes stationary) the action. To provide a powerful and elegant way to derive the equations of motion. To lay the groundwork for Hamiltonian mechanics, which is essential in both classical and quantum physics.This is used to adapt Hamiltons principle to systems that have non-holonomic constraints. Thus allowing the powerful tool of Hamiltons principle to be used in a larger set of problemsTo provide a general and powerful method for deriving the equations of motion for any dynamical system.To simplify the analysis of systems with constraints by using generalized coordinates.To express the equations of motion in terms of the Lagrangian, which is a function of the system's generalized coordinates and velocities.

To eliminate the need to explicitly consider constraint forces. To provide a versatile tool that can be applied to a wide range of mechanical systems.

In essence, these concepts work together to:

- Provide increasingly abstract and powerful ways to describe and analyze the motion of dynamical systems.
- Simplify the handling of constraints.
- Lay the foundation for advanced mechanics and related fields.

STRUCTURE:

4.5 Technical Terms

4.6 Self-Assessment Questions

4.7 Suggested Readings

4.1 DEDUCTION OF HAMILTON'S PRINCIPLE FROM D'ALEMBERT'S PRINCIPLE:

The variation of the potential energy V(r) may be expressed in terms of variations of the coordinates r_i

$$\delta V = \sum_{i=1}^{n} \frac{\partial y}{\partial r_i} \delta r_i = \sum_{i=1}^{n} f_i \, \delta r_i$$

Where f_i are potential forces collocated with coordinates r_i. In cartesian coordinates, the variation of the kinetic energy $T(\dot{r})$

$T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{r_i}^2$

May be expressed in terms of variations of coordinate velocities $\dot{r}_i$

$\delta T = \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{r}_i} = \sum_{i=1}^{n} m \, \dot{r}_i \delta \dot{r}_i.$

For a system of n particle masses m_i acted on by n internal forces f_i of the potential V, D'Alembert's principle is

$\sum_{i=1}^{n} \frac{\partial T}{\partial \dot{r}_i} = \sum_{i=1}^{n} m_i \, \ddot{r}_i \delta r_i = 0$

Integrating D'Alembert's equation over a finite time period,

$$\int_{t_1}^{t_2} \sum_{\iota=1}^{n} m_\iota \ddot{r}_\iota \delta r_i \, \mathrm{dt} + \int_{t_1}^{t_2} \sum_{\iota=1}^{n} \ddot{f}_\iota \; \delta r_i \, \mathrm{dt} = 0$$

$$\sum_{i=1}^{n} \int_{t_1}^{t_2} m_i \frac{d}{dt} \dot{r_i} \;\; \delta r_i \, \mathrm{dt} + \int_{t_1}^{t_2} \delta V \, dt = 0$$

$$\sum_{i=1}^{n} [m_i \dot{r_i} \;\; \delta r_i \,|_{t_1}^{t_2} - \int_{t_1}^{t_2} m_i \frac{d}{dt} \dot{r_i} \;\; \delta r_i \, \mathrm{dt}] + \int_{t_1}^{t_2} \delta V \, dt = 0$$

$$- \int_{t_1}^{t_2} m_i \frac{d}{dt} \dot{r_i} \;\; \delta r_i \, \mathrm{dt} + \int_{t_1}^{t_2} \delta V \, dt = 0$$

$$- \int_{t_1}^{t_2} \delta T \;\; \mathrm{dt} + \int_{t_1}^{t_2} \delta V \, dt = 0$$

$$\delta \int_{t_1}^{t_2} (T - V) \, dt = 0 \qquad (1)$$

In this derivation, a variation of the coordinate motions δr(t) from t_1 to t_2 is considered. That is, $\delta r(t_1) = 0$ and $\delta r(t_2) = 0$, which eliminates the first term in the third line. The fourth line involves a transposition of the variation and the derivative (d(δr)/dt = δr). The last line is a statement of Hamilton's principle, which is presented formally in the next section. Note that kinetic energy and potential energy are scalar-valued quantities, invariant to changes in coordinate systems. So, while Hamilton's principle is derived here in the context of Cartesian coordinates, it applies to generalized coordinates as well.

4.2 MODIFIED HAMILTON'S PRINCIPLE:

Using the definition of H in the action,

$$\mathbf{I} = \int_{t_1}^{t_2}[\textstyle\sum_i p_i\dot{q}_i - H\ (q,p,t)]dt \qquad (2)$$

We want to vary I to obtain Hamilton's equations. Since in the Hamiltonian formulation of mechanics, the coordinates and momenta are on equal footing, we would like to vary them independently in varying I. These variables are however, not independent but must satisfy the constraints.

$$\dot{q}_i - \frac{\partial H}{\partial p_i} = 0 \qquad (3)$$

Which arise from the definitions of $p_i$and H. The trivial identitiesof the form

$A_i(q, p, t) = A_i(q, p,t)$.

We can vary I independently with respect to q_i and p_i if we introduce the constraint (3) into the action via Lagrange multipliers λ_i. The action becomes

$$I^* = \int_{t_1}^{t_2}\left[\sum_i p_i\dot{q}_i - H + \sum_i \lambda_i(\dot{q}_i - \frac{\partial H}{\partial p_i})\right]dt$$

$$I^* = \int_{t_1}^{t_2} f(q,\dot{q},p,\lambda)dt \qquad (4)$$

The equations of motion which are obtained using the Lagrange multiplier rule are then

$$\frac{\partial}{\partial t}\frac{\partial f}{\partial \dot{p}_i} - \frac{\partial f}{\partial p_i} = 0 \qquad (5)$$

$$\frac{\partial}{\partial t}\frac{\partial f}{\partial \dot{q}_i} - \frac{\partial f}{\partial q_i} = 0 \qquad (6)$$

$$\frac{\partial f}{\partial \lambda_i} = 0$$

$$\dot{q}_i - \frac{\partial H}{\partial p_i} = 0 \qquad (7)$$

In obtaining these equations of motion we have varied q_i, p_i and λ_i independently in accord with the multiplier rule. Equation (7) yields the equations of motion

$$\dot{q}_i - \frac{\partial H}{\partial p_i} - \sum_j \lambda_j (\frac{\partial^2 H}{\partial p_i \partial p_j}) = 0 \qquad (8)$$

For a nondegenerate system we must have

$$\text{Det}(\frac{\partial^2 H}{\partial p_i \partial p_j}) \neq 0 \qquad (9)$$

Which when combined with (7) implies that all Lagrange multipliers λ_i are zero,

$$\lambda_i = 0 \qquad (10)$$

This means that in the variational principle we will obtain the same results even if we ignore the constraint and vary the action (2) with respect to independent variation of q_i and $p_i$is exactly the modified Hamilton's principle. This shows that the usual assumption of ignoring the constrained implied by $\dot{q}_i = \frac{\partial H}{\partial p_i}$ is indeed justified by general results from the calculus of variations. Finally (5) and (6) with $\lambda_i = 0$ in f give Hamilton's equations of motion.

4.3 HAMILTON'S PRINCIPLE AND LAGRANGE'S EQUATIONS:

Since the state of the particle is specified by its location and velocity at a particular time, we look for some function of those variables to work with. Then we look for a general principle involving this function that tells us how the external world influences the particle's state.

It was recognized early on that cartesian coordinate axes are not the only way to specify location. For the curved track referred to above it would be helpful to have a coordinate that just told us how far the particle has moved along the track. Such specifications are called generalized coordinates, denoted by q_i. There are as many of these as there are independent ways for the particle to move; these ways are called degrees of freedom. Each coordinate has its corresponding velocity $\dot{q}_i = \frac{\partial p_i}{\partial t}$. Then the function we seek will be called L(qi ,q!i ,t). (For brevity, we will often omit the subscripts i.)

The general principle we need was given in its final form by Hamilton, and is often called the principle of least action. The term "action" refers to the integral of L over time:

$$J = \int_{t_1}^{t_2} L(\, q, \dot{q} \,, t)\, dt.$$

Here the limits are two times at which the particle has two different states. We imagine that these times and the corresponding states are fixed, but that we can vary both q $and \dot{q}$during the time in between, making the particle follow different paths, so that J is varied. Calling these variations δq , $\delta\dot{q}$, and δJ , we have

$$\delta J = \int_{t_1}^{t_2} L(\, q + \delta q, \dot{q} + \delta\dot{q} \,, t) dt - \int_{t_1}^{t_2} L(\, q, \dot{q} \,, t)\, dt.$$

Hamilton's principle says that for the actual motion of the particle, δJ= 0 to first order in the variations δq and $\delta\dot{q}$. That is, the actual motion of the particle is such that small variations do not change the action.

Now by Taylor's theorem we can write to 1st order

$$L(\, q + \delta q,\; \dot{q} + \delta\dot{q}\,, t) \approx L(\, q, \dot{q}\,, t) + \frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}\,,$$

Where the partial derivatives are evaluated for $\delta q = \delta\dot{q}$ = 0. Thus we find

$$\delta J = \int_{t_1}^{t_2}[\frac{\partial L}{\partial q}\delta q + \frac{\partial L}{\partial \dot{q}}\delta\dot{q}]\, dt.$$

Since $\dot{q}$ = dq/dt we have $\delta\dot{q}$ = d(δq)/dt , so the 2nd term in the integral is

$$\int_{t_1}^{t_2}\frac{\partial L}{\partial \dot{q}}\frac{d(\delta q)}{dt}\, dt,$$

and we integrate by parts to convert this to

$$\frac{\partial L}{\partial \dot{q}}\delta q|_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d}{dt}\frac{\partial(\delta q)}{\partial \dot{q}}.\delta q\, dt.$$

Because the states at the initial and final times are fixed, δq vanishes at both times, sothe first term above is zero.

We have then

$$\delta J = \int_{t_1}^{t_2}[\frac{\partial L}{\partial q} - \frac{d}{dt}(\frac{\partial L}{\partial \dot{q}})]\delta q\, dt = 0.$$

Since δq is arbitrary, the quantity in must vanish, so we have finally

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) \qquad (11)$$

The becomes a differential equation (2nd order in time) to be solved. It is the equation of motion for the particle, and is called Lagrange's equation. The function L is called the Lagrangian of the system. Here we need to remember that our symbol q actually represents a set of different coordinates. Because there are as many q's as degrees of freedom, there are that many equations represented by Eq (11).

We have used the D'Alembert's principle to deduce Lagrange's equations. This principle uses the idea of virtual work and 'steins from Newton's second law of motion. These Lagrange's equations can be derived by an entirely different way namely Hamilton's variational principle.

This principle states that for a conservative holonomic system, its motion from time t_1 to time t_2 is such that the line integral known as (**action** or **action integral)**

$$S = \int_{t_1}^{t_2} L\, dt \qquad (12)$$

with T- V has stationary (extremum) value for the correct path of the motion.

The quantity **S** is called as **Hamilton's principal function.** The principle may be expressed as

$$\delta \int_{t_1}^{t_2} L\,dt = 0$$

where δ is the variation symbol

Properties of the Lagrangian:

So Hamilton's principle has given us Eq (11) for the Lagrangian. What do we know about L beyond the variables it depends on? We assume we are in an inertial reference frame. Then all coordinate axes are equivalent, so L must be a scalar. And our choice of when to start the clocks is arbitrary, so L cannot depend explicitly on t.

Beyond that we can make some reasonable requirements. Suppose we have two systems A and B separated by large distances so they do not interact with each other. Then the Lagrangian for this composite system must consist of separate parts for each, i.e., L(A + B) = L(A) + L(B) . Furthermore, multiplying L by some constant would change nothing in the equations so far. Choice of that constant simply involves choosing a system of units.

Another thing that does not change the physical content of the Lagrangian is adding to it the total time derivative of a function of q and t. Suppose we define

$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + df(q,t)/dt$.

Then since

$$\int_{t_1}^{t_2} \frac{df}{dt}\,dt = f(t_2) - f(t_1)$$

for the action we have

$$J' = J + f(t_2) - f(t_1)$$

The terms in f evaluated at the endpoints do not change when we perform the variation, so $\delta J' = \delta J$. The two Lagrangians give the same variation and are thus equivalent in physical content.

Now we take the simplest system, a particle moving without any interaction with the external world. We know its velocity is constant (the 1st law). Since all points in space are equivalent for such a particle, L cannot depend on its position x. It must therefore depend only on the velocity v. But it is a scalar, so it can depend only onv^2. We have that$\delta L/\delta x_i = 0$, so by Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v_i}\right) = 0$$

showing that $\frac{\partial L}{\partial v_i}$is constant. But

$$\frac{\partial L}{\partial v_i} = \frac{\partial L}{\partial v^2}\frac{\partial v^2}{\partial v_i} = 2\frac{\partial L}{\partial v^2}.v_i.$$

we know that v_i is constant. This means

$\frac{\partial L}{\partial v^2}$= const.

We conclude that L = (const).v^2. We choose the constant to bem/2 and have L = 1/2 mv^2,

the kinetic energy T of the particle.

Now we introduce interactions of the particle with its environment, In Newtonian mechanics these are described by forces, the connection to the motion being given by the 2nd law. We try to introduce these into to the Lagrangian by adding a term to the one we already have.

Suppose the interaction term in L does not depend explicitly on the particle's velocity. Then we will have$\frac{\partial L}{\partial v_i} = \frac{\partial T}{\partial v_i}$= mv_i, and Lagrange's equation becomes

$$\frac{\partial}{\partial t}(mv_i) - \frac{\partial L}{\partial x_i} = 0\,,$$

or

$$m\ddot{x}_i = \frac{\partial L}{\partial x_i}.$$

For this to give us the 2nd law we need the right side to be the force. We know this to be given by $\frac{\partial U(x)}{\partial x_i}$, where U is the potential energy function for the force. We are thus led to the final form for the Lagrangian:

$$L(x_i\,,\, v_i,t) = T - U(x_i,t). \qquad (13)$$

The possible dependence of U on t might arise if the locations of objects with which our particle interacts are changing with time in a known way. In most of our cases, U will not depend on t.

4.4 SUMMARY:

Here's a concise summary of the relationships between D'Alembert's principle, Hamilton's principle, modified Hamilton's principle, and Lagrange's equations:

- **D'Alembert's Principle:**

  This principle states that the virtual work done by the impressed forces and the inertial forces in a system is zero. It's a generalization of static equilibrium to dynamics, expressed as a sum of forces times virtual displacements. It provides a foundation for deriving equations of motion.

- **Hamilton's Principle:**

  Hamilton's principle is a variational principle that states that the actual path taken by a system between two points in time is the one that minimizes the action integral. The action [1] is the integral of the Lagrangian (difference between kinetic and potential energy) over time. Hamilton's principle can be derived from D'Alembert's principle by integrating over time and introducing the concept of the Lagrangian.

- **Modified Hamilton's Principle:**

  This principle extends Hamilton's principle to include non-conservative forces. It modifies the action integral to account for dissipative or external forces that do not arise from a potential.

- **Lagrange's Equations:**

  Lagrange's equations are derived from Hamilton's principle. They provide a set of differential equations that describe the motion of a system in terms of generalized coordinates and the Lagrangian. These equations are particularly useful for systems with constraints, as they eliminate the need to explicitly consider constraint forces.

Essentially, D'Alembert's principle is a stepping stone to Hamilton's principle, which, through variational calculus, leads to Lagrange's equations. The modified version of Hamilton's principle allows for the inclusion of non-conservative forces.

4.5 TECHNICAL TERMS:

D'Alembert's Principle, Hamilton's principle, Lagrange's equations.

4.6 SELF-ASSESSMENT QUESTIONS:

1) Deduce of Hamilton’s principle from D’Alembert’s principle.
2) Write modified Hamilton’s principle.
3) Write the Hamilton’s principle and Lagrange’s equations.

4.7 SUGGESTED READINGS:

1) Classical Mechanics by H.Goldstein.
2) Fundamentals of Classical Mechanics by J.C. Upadhyaya.
3) Classical Mechanics by G. Aruldhas, PHI Publishers.
4) The Theory of relativity and applications, Allen Rea.

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LESSON-5

HAMILTON'S PRINCIPLE

5.0 AIM AND OBJECTIVES:

To learn about

- Lagrange's Equation from Hamilton's Principle
- Lagrange's Equation for Non-Conservative
- Non-Holonomic System

To establish a rigorous and elegant derivation of Lagrange's equations using the variational principle of Hamilton, to demonstrate the fundamental connection between the Lagrangian formulation and the principle of least action and to provide a deeper understanding of the theoretical foundations of Lagrangian mechanics. To show that Lagrange's equations are a direct consequence of minimizing the action integral, which is the integral of the Lagrangian over time. To use the calculus of variations to derive Lagrange's equations from the condition that the action is stationary. To illustrate the power of variational principles in deriving equations of motion. And to show that Hamilton's principle is a more fundamental way to describe the motion of a system, from which Lagrange's equations can be derived.To extend the applicability of Lagrange's equations to systems that include non-conservative forces (e.g., friction, drag) and non-holonomic constraints (e.g., rolling without slipping).To provide a method of analyzing a wider variety of realistic mechanical systems. To adapt the Lagrangian formalism to handle situations where the usual assumptions of conservative forces and holonomic constraints do not hold.To modify Lagrange's equations to incorporate the effects of non-conservative forces by introducing generalized forces. To develop techniques for handling non-holonomic constraints, which cannot be expressed as simple equations relating the generalized coordinates and to derive equations of motion that accurately describe the behavior of systems with dissipative forces and constraints that limit the possible virtual displacements.To allow the use of Lagrange's equations to solve real world problems that include friction, or other non conservative forces, and also systems that include constraints that are inequalities and to be able to create a generalized form of Lagrange's equations that can be used in a wide range of mechanical systems.

STRUCTURE:

5.5 Technical Terms

5.6 Self-Assessment Questions

5.7 Suggested Readings

5.1 LAGRANGE'S EQUATION FROM HAMILTON'S PRINCIPLE:

The Lagrangian ***L*** is a function of generalized coordinates q_k, and generalized velocities $\dot{q}_k$ and time *t, i.e.,*

$$\boldsymbol{L} = \boldsymbol{L}\,(q_1, q_2, \ldots, q_k, \ldots, q_n, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k, \ldots, \dot{q}_n, t)$$

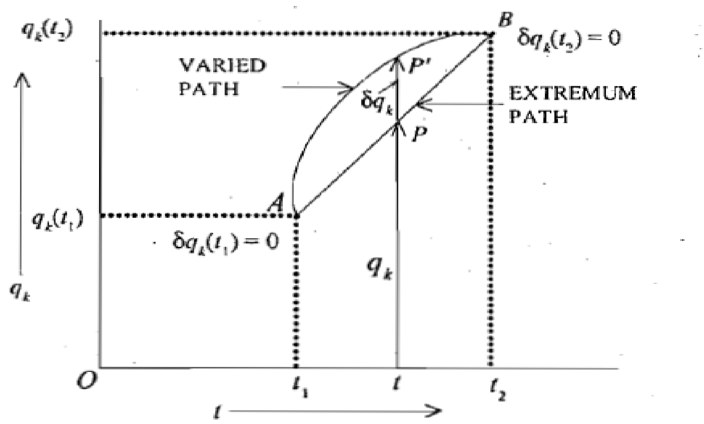

Fig. 1.5:δ variation - extremum path

If the Lagrangian does not depend on time ***t*** explicitly, then the variation δL can be written as

$$\delta L = \sum_{k=1}^{n} \frac{\partial L}{\partial q_k} \delta q_k + \sum_{k=1}^{N} \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \qquad (1)$$

Integrating both sides from $t = t_1$ *to* $t = t_2$, we get

$$\int_{t_1}^{t_2} \delta L \, dt = \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt$$

But in view of the Hamilton's principle

$$\delta \int_{t_1}^{t_2} L \, dt = 0$$

Therefore, $$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k} \delta q_k \, dt + \int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt = 0 \qquad (2)$$

where $\delta \dot{q}_k = \frac{d}{dt}(\delta q_k)$

Integrating by parts the second term on the left hand side of eq. (2), we get

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial \dot{q}_k} \delta \dot{q}_k \, dt = \sum_k \left[\frac{\partial L}{\partial \dot{q}_k} \delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \delta q_k \, dt \qquad (3)$$

At the end points of the path at the times $t_1$and t_2, the coordinates must have definite values $q_k(t_1)$and $q_k(t_2)$respectively, *i.e.*, $\delta q_k(t_1) = \delta q_k(t_2)$= 0 (Fig. 1.5) and hence

$$\sum_k \left[\frac{\partial L}{\partial q_k}\delta q_k\right]_{t_1}^{t_2} = 0$$

Therefore, eq. (2) takes the form

$$\int_{t_1}^{t_2} \sum_k \frac{\partial L}{\partial q_k}\delta q_k\, dt - \int_{t_1}^{t_2} \sum_k \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right)\delta q_k\, dt = 0$$

$$\sum_k \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k}\right]\delta q_k\, dt = 0 \quad (3)$$

For holonomic system, the generalized coordinates δq_k are independent of each other. Therefore, the coefficient of each δq_k must vanish, *i.e.*,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} = 0 \quad (4)$$

where k =*1*, 2,...., *n* are the generalized coordinates.

Eqs. (4) are the *Lagrange's equations of motion*

5.2 LAGRANGE'S EQUATION FOR NON-CONSERVATIVE AND NON-HOLONOMIC SYSTEM:

We deduced Hamilton's principle from D'Alernbert's principle for conservative forces. If the forces are not conservative,

$$\frac{d}{dt}\left[\sum_i m_i \dot{r_i} \cdot \delta r_i\right] = \delta T + \delta W \quad (5)$$

where $\delta T = \delta \sum \frac{1}{2} m_i v_i^2$and $\delta W = \sum F_i \cdot \delta r_i$ = virtual work done

The integration of (5) from $t = t_1$to $t = t_2$ with the condition $\delta r_i(t_1) = \delta r_i(t_2) = 0$ at the end points, we get

$$\int_{t_1}^{t_2} \delta[T + W]\, dt = 0 \; or \; \delta \int_{t_1}^{t_2} [T + W]\, dt = 0 \quad (6)$$

Eq. (6) is known as *extended Hamilton's principle.* Here F_i are the non-conservative forces. We can write as

$$\delta W = \sum_i F_i \cdot \delta r_i = \sum_{i,k} F_i \cdot \frac{\partial r_i}{\partial q_k}\delta q_k = \sum_k G_k \delta q_k \quad (7)$$

where $G_k$are the components of generalized force

Thus, the extended Hamilton's principle (6) gives

$$\delta \int_{t_1}^{t_2} T\, dt + \int_{t_1}^{t_2} \sum_k G_k \delta q_k\; dt = 0 \quad (8)$$

Kinetic energy *T* in general is function of $q_k$and $\dot{q_k}$and hence

$$\delta\int_{t_1}^{t_2} T(q_k, \dot{q_k})dt = \int_{t_1}^{t_2} \delta T(q_k, \dot{q_k})dt = \int_{t_1}^{t_2}\sum_k \left[\frac{\partial T}{\partial q_k}\delta q_k + \frac{\partial T}{\partial \dot{q_k}}\delta \dot{q_k}\right]dt$$

$$= \int_{t_1}^{t_2}\sum_k \frac{\partial T}{\partial q_k}\delta q_k\, dt + \sum_k \left[\frac{\partial T}{\partial \dot{q_k}}\delta q_k\right]_{t_1}^{t_2} - \int_{t_1}^{t_2}\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q_k}}\right]\delta q_k dt$$

$$= \int_{t_1}^{t_2}\sum_k \left[\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q_k}}\right]\right]\delta q_k\, dt\, [\because \delta q_k(t_1) = \delta q_k(t_2) = 0] \quad (9)$$

Thus eq. (8) is

$$\int_{t_1}^{t_2}\sum_k \left[\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q_k}}\right] + G_k\right]\delta q_k\, dt = 0 \quad (10)$$

Since the constraints are holonomic, all $\delta q_k$are independent and hence the integral will vanish, if

$$\frac{\partial T}{\partial q_k} - \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q_k}}\right] + G_k = 0 \;\; or \;\; \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q_k}}\right] - \frac{\partial T}{\partial q_k} = G_k \quad (11)$$

These are the Lagrangian equations for holonomic and non-conservative system.

5. 3 LAGRANGE'S EQUATIONS OF MOTION FOR NON-HOLONOMIC SYSTEMS:

In the derivation of Lagrange's equations from D'Alembert's principle or Hamilton's principle, we need the requirement of holonomic constraints at the final step, when the variations 8q_k are considered to be independent of each other. In case of non-holonomic systems, the generalized coordinates are not independent of each other. However, we can treat certain types of non-holonomic systems for which the equations of constraint can be put in the form

$$\sum_k a_{lk}dq_k + a_{lt}dt = 0 \quad (12)$$

These equations of constraints connect the differentials $dq_k's$ by linear relations. For each *l* there is one equation and we assume that there are *m* such, equations for *l*= *1,* 2,...., m

In case of δ -variation, the virtual displacements $\delta q_k$are taken at constant times and hence the *m* equations of constraints, consistent for virtual displacements, are

$$\sum_k a_{lk}\delta q_k = 0 \quad (13)$$

Eq. (13) now can be used to reduce the number of virtual displacements to independent ones. The procedure applied for this purpose is called *Lagrange's method of undetermined multipliers.*

If eq. (13) is valid, then the multiplication of this equation by λ_l an undetermined quantity, yields

$$\lambda_l\sum_k a_{lk}\delta q_k = 0 \;\; or \sum_k \lambda_l a_{lk}\delta q_k = 0 \quad (14)$$

where $\lambda_l(1,2,...,m)$ are undetermined quantities and they are functions in general of the coordinates and time: Summing eq. (40) over *l* and then integrating the sum with respect to time from $t = t_1$ to $t = t_2$ we get

$$\int_{t_1}^{t_2} \sum_{kl} \lambda_l a_{lk} \delta q_k dt = 0 \tag{14}$$

We assume the Hamilton's principle

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \tag{15}$$

to hold for the non-holonomic system. This implies that

$$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right)\right] \delta q_k\, dt = 0 \tag{16}$$

adding (14) and (16), we obtain

$$\int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) + \sum_l \lambda_l a_{lk}\right] \delta q_k\, dt = 0 \tag{17}$$

Still, all $\delta q_k{}'s$ $(k = 1,2,...,n)$ are not independent of each other. First *n* — *m* of these $\delta q_k{}'s$ may be chosen independently and the last *m* of these $\delta q_k{}'s$are then fixed by the eq. (13).

Till now the values of $\lambda_l$have not been specified. We choose the $\lambda_l{}'s$ such that

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) + \sum_{l=1}^{m} \lambda_l a_{lk} = 0 \tag{18}$$

where *k* = *n- m + 1, n- m* + 2, ..., *n.* Thus eqs. (18) will determine *m* values of $\lambda_l$and then eq. (17) can be written as

$$\int_{t_1}^{t_2} \sum_{k=1}^{n-m} \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) + \sum_l \lambda_l a_{lk}\right] \delta q_k\, dt = 0 \tag{19}$$

where the *8q$_k$'s (k* =1, 2, ..., *n - in),* involved, are independent ones. Therefore, for the integrand in (19) to vanish

$$\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) + \sum_{l=1}^{m} \lambda_l a_{lk} = 0 \tag{20}$$

which is *n-m* equations for *k* =1, 2, ..., *n- m.*

Adding eqs. (18) and (20), we get the complete set of the Lagrange's equations for the non-holonomic system, *i.e.*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} = \sum_{l=1}^{m} \lambda_l a_{lk} \tag{21}$$

where *k* = 1, 2, ...*n.*

This gives us *n* equations, but there is *n* + *m* unknowns, *n* coordinates q_k and *m* Lagrange's multipliers. The remaining *m* unknown $q_k{}'s$ are determined from m equations of constraints eq. (12), written in the following form of ***m*** first-order differential equations

$$\sum_k a_{lk} \dot{q_k} + a_{lt} = 0 \tag{22}$$

5.4 SUMMARY:

Lagrange's Equation from Hamilton's Principle:

- This derivation demonstrates that Lagrange's equations arise directly from the principle of least action, which states that a system's path minimizes the action integral. It uses variational calculus to show that Lagrange's equations are a consequence of this minimization, establishing a fundamental link between Lagrangian mechanics and variational principles.

Lagrange's Equation for Non-Conservative, Non-Holonomic Systems:

- This extension adapts Lagrange's equations to handle real-world complexities. It incorporates non-conservative forces (like friction) through generalized forces and addresses non-holonomic constraints (like rolling without slipping) that cannot be expressed as simple equations. This modification allows the Lagrangian formalism to be applied to a broader range of mechanical systems, including those with dissipative forces and complex constraints.
- In this lesson we learn about the complete derivation of Lagrange's Equation from Hamilton's Principle and Lagrange's Equation for Non-Conservative, Non-Holonomic Systems.

5.5 TECHNICAL TERMS:

Lagrange's Equation from Hamilton's Principle and Lagrange's Equation for Non-Conservative, Non-Holonomic Systems

5.6 SELF-ASSESSMENT QUESTIONS

1) Derive Lagrange's Equation from Hamilton's Principle.
2) Lagrange's Equation for Non-Conservative, Non-Holonomic Systems

5.7 SUGGESTED READINGS:

1) Classical Mechanics by H.Goldstein.
2) Fundamentals of Classical Mechanics by J.C. Upadhyaya.
3) Classical Mechanics by G. Aruldhas, PHI Publishers.
4) The Theory of relativity and applications, Allen Rea.

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LESSON-6

APPLICATIONS OF HAMILTON PRINCIPLE

6.0 OBJECTIVES:

To learn about

- Simple applications of Hamilton principle- linear harmonic oscillator
- Simple pendulum, Δ-variation
- Principle of Least Action

To demonstrate the application of Hamilton's principle to a fundamental and well-understood system: the linear harmonic oscillator. To illustrate how Hamilton's principle can be used to derive the equations of motion for a simple system. To show the elegance and effectiveness of the Lagrangian and Hamiltonian approach, compared to newtonian methods. To express the Lagrangian for the linear harmonic oscillator in terms of its kinetic and potential energies. To apply Hamilton's principle to find the path that minimizes the action integral. To derive the well-known equation of motion for the linear harmonic oscillator. To reinforce the understanding of how to use Hamilton's principle. To apply Hamilton's principle and the concept of variations (specifically, Δ-variation) to derive the equation of motion for a simple pendulum. To illustrate the use of generalized coordinates and the Lagrangian formalism in a system with constraints. To show how to apply the calculus of variations to a real world problem. To express the Lagrangian for the simple pendulum in terms of its angular displacement. To perform the Δ-variation of the action integral. To derive the equation of motion for the simple pendulum. To show how to handle the constraint of the pendulums length within the lagrangian framework. To understand the fundamental principle that governs the dynamics of physical systems. To establish the connection between the Lagrangian formulation and the concept of minimizing the action. To show how the path a system takes through configuration space, is the path that minimizes the action. To define the action integral and its significance. To explain that the actual path taken by a system is the one that makes the action integral stationary (usually a minimum).To demonstrate the relationship between the principle of least action and Hamilton's principle. To show how this principle can be used to derive the equations of motion for a system. To show the connection between this classical principle, and its more modern applications in quantum mechanics.

STRUCTURE:

6.1 Simple application of the Hamilton principle- linear harmonic oscillator

6.2 Simple Pendulum

6.3 Δ-Variation

6.1 SIMPLE APPLICATION OF THE HAMILTON PRINCIPLE-LINEAR HARMONIC OSCILLATOR:

Linear Harmonic Oscillator The linear harmonic oscillator is described by the Schrodinger equation

$$i\hbar\,\partial_t\,\psi(x,t) \;=\; \hat{H}\,\psi(x,t) \qquad (1)$$

for the Hamiltonian

$$\hat{H} \;=\; -\,\frac{\hbar^2}{2m}\,\frac{\partial^2}{\partial x^2} \;+\; \frac{1}{2}m\,\omega^2 x^2 \qquad (2)$$

It comprises one of the most important examples of elementary Quantum Mechanics. There are several reasons for its pivotal role. The linear harmonic oscillator describes vibrations in molecules and their counterparts in solids, the phonons. Many more physical systems can, at least approximately, be described in terms of linear harmonic oscillator models. However, the most eminent role of this oscillator is its linkage to the boson, one of the conceptual building blocks of microscopic physics. For example, bosons describe the modes of the electromagnetic field, providing the basis for its quantization. The linear harmonic oscillator, even though it may represent rather non-elementary objects like a solid and a molecule, provides a window into the most elementary structure of the physical world. The most likely reason for this connection with fundamental properties of matter is that the harmonic oscillator Hamiltonian (2) is symmetric in momentum and position, both operators appearing as quadratic terms. The important role of the harmonic oscillator certainly justifies an approach from two perspectives, i.e., from the path integral (propagator) perspective and from the Schrodinger equation perspective. The path integral approach gave us a direct route to study time-dependent properties, the Schrodinger equation approach is suited particularly for stationary state properties. Both approaches, however, yield the same stationary states and the same propagator, as we will demonstrate below. The Schrodinger equation approach will allow us to emphasize the algebraic aspects of quantum theory. This Section will be the first in which an algebraic formulation will assume center stage. In this respect the material presented provides an important introduction to later Sections using Lie algebra methods to describe more elementary physical systems. Due to the pedagogical nature of this Section we will link carefully the algebraic treatment with the differential equation methods used so far in studying the Schrodinger equation description of quantum systems.

We consider first the stationary states of the linear harmonic oscillator and later consider the propagator which describes the time evolution of any initial state. In the framework of the Schrodinger equation the stationary states are solutions of (1) of the form

$$\psi(x,t) \;=\; \exp(-iEt/\hbar)\phi_E(x,t) \qquad (3)$$

where

$$\hat{H}\,\phi_E(x) \;=\; E\,\phi_E(x) \qquad (4)$$

Due to the nature of the harmonic potential which does not allow a particle with finite energy to move to arbitrarily large distances, all stationary states of the harmonic oscillator must be bound states and, therefore, the natural boundary conditions apply

$$\lim_{x\to\pm\infty} \phi_E(x) \;=\; 0 \qquad (5)$$

Equation (3) can be solved for any E ∈ R, however, only for a discrete set of E values can the boundary conditions (5) be satisfied. In the following algebraic solution of (3) we restrict the Hamiltonian Hˆ and the operators appearing in the Hamiltonian from the outset to the space of functions

$$\mathcal{N}_1 \;=\; \{f:\mathbb{R}\to\mathbb{R},\, \mho \in \mathbb{C}_\infty,\, \lim_{\curvearrowleft\to\pm\infty} \mho(\curvearrowleft) \;=\; \nVdash\} \qquad (6)$$

where C_∞ denotes the set of functions which together with all of their derivatives are continuous. It is important to remember this space restriction, in which the operators used below, act. We will point out explicitly where assumptions are made which built on this restriction. If this restriction would not apply and all functions f : R → R would be admitted, the spectrum of Hˆ in (3) would be continuous and the eigen functions φE(x) would not be normalizable.

6.2 SIMPLE PENDULUM:

Here, θ is the generalized coordinate. The velocity of the ball is lθ' acting perpendicular to OP where l is the length of the string of the pendulum (see, Fig 6.1). A simple pendulum oscillating in a vertical plane constitutes a conservative holonomic dynamical system. Here, Kinetic Energy = T = $(1/2)ml^2\dot{\theta}^2$, m is the mass of the ball.

Potential Energy = V = mgh = mgl(1- cosθ).

Therefore, the Lagrangian of the system is L = T - V = $(1/2)ml^2\dot{\theta}^2$, - mgl(1- cosθ).

Since the system is scleronomous and conservative,

H = T + V = $(1/2)ml^2\dot{\theta}^2$ + mgl(1 - cosθ).

Generalized momentum $p_\theta = \partial L/\partial\dot{\theta} = ml^2\dot{\theta}$ or $\dot{\theta} = p_\theta/ml^2$.

H = (1/2)ml^2(p_θ/ml^2)² + mgl(1 - cosθ) = (1/2)(p_θ/ml^2)+ mgl(1-cosθ) = H(θ, p_θ).

Hamilton's equations of motion are given by:

$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = p_\theta / ml^2$, i.e., $p_\theta = ml^2\,\dot{\theta}$ and

$$\dot{p_\theta} = -\frac{\partial H}{\partial \theta} = -mglsin\theta$$

Therefore $ml^2\ddot{\theta} = -mglsin\theta$ or $\ddot{\theta} = -\frac{gsin\theta}{l} \approx \frac{g}{l}\theta$ where θ is very small.

Time period is given by $2\pi\sqrt{\frac{l}{g}}$, g is the acceleration due to gravity.

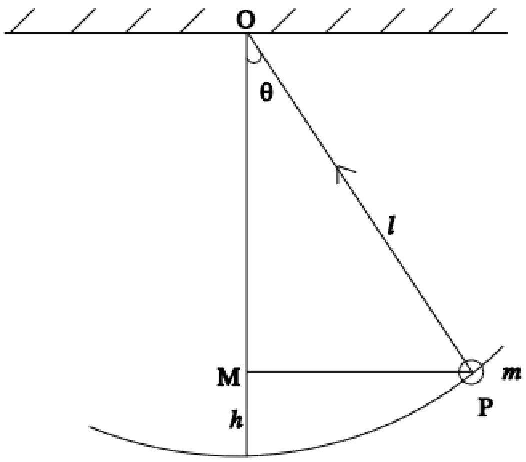

Fig: 6.1

6.3 Δ-VARIATION:

The δ-variation that we considered is refers to the variation in a quantity at the same instant of time. The varied path in configuration space always terminates at the end-points $t_1 and\ t_2$ such that $\delta q_i(t_1) = \delta q_i(t_2) = 0$. The Δ-variation, a more general type of variation of the path of the system, is one in which time as well as position co-ordinates vary in the configuration space. At the end-points of the path, the position co-ordinates are all kept fixed while the time co-ordinate may change. Fig. 6.2 illustrates the Δ-variation of a co-ordinateq_i in the configuration space.

Let ABC be the actual path and $A', B'\ and\ C'$the varied path. The end-points of the path A and Ctake the positionsA' and C' after timeΔt, such that there is on change in position co-ordinates, ie, $\Delta q_i(1) = \Delta q_i(2) = 0$. The point Bon theactual path now goes over to the point B'. on the varied path such that

$$q_i' = q_i + \Delta q_i = q_i + \delta q_i + \dot{q}_i\,\Delta t \qquad (8)$$

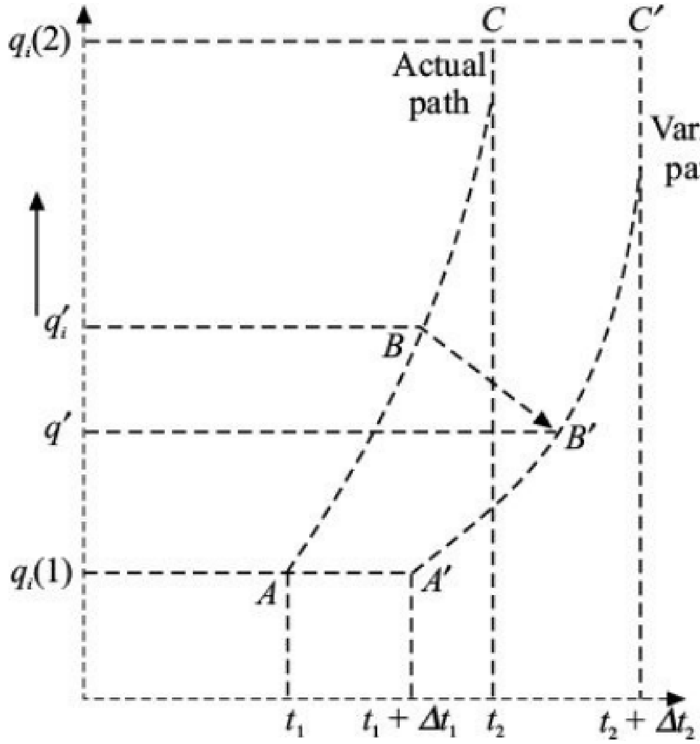

Fig. 6.2: Illustration of Δ-variation in Configuration Space

The Δ-variation of any function $f = f(q, \dot{q}, t)$ is given by

$$\Delta f = \sum_i \left(\frac{\partial f}{\partial q_i} \Delta q_i + \frac{\partial f}{\partial \dot{q}_i} \Delta \dot{q}_i \right) + \frac{\partial f}{\partial t} \Delta t$$

Using Eq. (6.95)

$$\Delta f = \sum_i \frac{\partial f}{\partial q_i} \left(\delta q_i + \dot{q}_i \Delta t \right) + \sum_i \frac{\partial f}{\partial \dot{q}_i} \left(\delta \dot{q}_i + \ddot{q}_i \Delta t \right) + \frac{\partial f}{\partial t} \Delta t$$

Rearranging

$$\Delta f = \sum_i \left(\frac{\partial f}{\partial q_i} \delta q_i + \frac{\partial f}{\partial \dot{q}_i} \delta \dot{q}_i \right) + \left[\sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial \dot{q}_i} \ddot{q}_i \right) + \frac{\partial f}{\partial t} \right] \Delta t$$

$$= \delta f + \Delta t \frac{df}{dt} \tag{9}$$

Thus, the δ and Δ operations are connected by the relation

$$\Delta = \delta + \Delta t \frac{d}{dt} \tag{10}$$

6.4 PRINCIPLE OF LEAST ACTION:

According to the principle of least action

$$\Delta \int_{t_1}^{t_2} \sum_k p_k \dot{q_k} dt = 0 \qquad (11)$$

where the quantity $W = \int_{t_1}^{t_2} \sum_k p_k \dot{q_k}\, dt$ is sometimes called *abbreviated action.*

Eq. (11) was established by Maupertuis (1668-1759) and therefore it is usually referred Maupertuis principle of least action.

Proof: Let us consider Hamilton's principle function (or action integral) *S*, given by

$$S = \int_{t_1}^{t_2} L dt \qquad (12)$$

The Δ-variation of S is

$$\Delta S = \Delta \int_{t_1}^{t_2} L dt = \left[\delta + \Delta t \frac{d}{dt}\right] \int_{t_1}^{t_2} L dt$$

$$= \delta \int_{t_1}^{t_2} L dt + \int_{t_1}^{t_2} \Delta t\, d(L) = \delta \int_{t_1}^{t_2} L dt + [L \Delta t]_{t_1}^{t_2}$$

$$= \int_{t_1}^{t_2} \delta L dt + [L\Delta t]_{t_1}^{t_2} [\because \delta(dt) = 0]$$

$$= \int_{t_1}^{t_2} \sum_k \left[\frac{\partial L}{\partial q_k}\, \delta q_k + \frac{\partial L}{\partial \dot{q_k}}\, \delta \dot{q_k}\right] dt + [L\Delta t]_{t_1}^{t_2} \qquad (13)$$

In the present case $\delta q_k \neq 0$ at the end points, hence of $\delta \int_{t_1}^{t_2} L dt$ is not equal to zero.

Now, according to Lagrange's equations, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) - \frac{\partial L}{\partial q_k} = 0 \; or \; \frac{\partial L}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q_k}}\right) \qquad (14)$$

Also $$\partial \dot{q_k} = \frac{d}{dt}[\delta q_k] \qquad (15)$$

Using (14) and (15), the quantity in the first term of eq. (13) is

$$\frac{\partial L}{\partial q_k}\, \delta q_k + \frac{\partial L}{\partial \dot{q_k}}\, \delta \dot{q_k} = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q_k}}\right] \delta q_k + \frac{\partial L}{\partial \dot{q_k}} \frac{d}{dt}[\delta q_k] = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q_k}} \delta q_k\right] = \frac{d}{dt}[p_k\, \delta q_k] \qquad (16)$$

But in view of $\Delta -$ operation equation

$$\Delta q_k = \delta q_k + \Delta t \frac{dq_k}{dt} \; or \; \delta q_k = \Delta q_k - \Delta t\, \dot{q_k} \; or \; p_k \delta q_k = p_k \Delta q_k - p_k \dot{q_k} \Delta t \qquad (17)$$

Hence $$\frac{\partial L}{\partial q_k}\, \delta q_k + \frac{\partial L}{\partial \dot{q_k}}\, \delta \dot{q_k} = \frac{d}{dt}[p_k \Delta q_k] - \frac{d}{dt}[p_k \dot{q_k} \Delta t] \qquad (18)$$

Thus eq. (12) is

$$\Delta S = \Delta\int_{t_1}^{t_2} L dt = \int_{t_1}^{t_2}\sum_k\left[\frac{d}{dt}[p_k\Delta q_k] - \frac{d}{dt}[p_k\dot{q}_k\Delta t]\right]dt + [L\Delta t]_{t_1}^{t_2}$$

$$= \sum_k\int_{t_1}^{t_2}\left[\frac{d}{dt}(p_k\Delta q_k) - \frac{d}{dt}(p_k\dot{q}_k\Delta t)\right] + [L\Delta t]_{t_1}^{t_2}$$

$$= \sum_k[p_k\Delta q_k]_{t_1}^{t_2} - \sum_k[p_k\dot{q}_k\Delta t]_{t_1}^{t_2} + [L\Delta t]_{t_1}^{t_2} \quad (19)$$

As $\Delta q_k = 0$ at the end points, $[p_k\Delta q_k]_{t_1}^{t_2} = 0$

Therefore equation (19) is

$$\Delta\int_{t_1}^{t_2} L dt = \left[\left(L - \sum_k p_k\dot{q}_k\right)\Delta t\right]_{t_1}^{t_2}$$

or $$\Delta\int_{t_1}^{t_2} L dt = -[H\Delta t]_{t_1}^{t_2} \qquad [\because H = \textstyle\sum_k p_k\dot{q}_k - L] \quad (20)$$

Now, if we restrict to systems for which $\frac{\partial H}{\partial t} = 0$and to variations for which H remains constant (conservative systems), then

$$\Delta\int_{t_1}^{t_2} H dt = \int_{t_1}^{t_2} H d(\Delta t) = [H\Delta t]_{t_1}^{t_2} \quad (21)$$

Substituting for $[H\Delta t]_{t_1}^{t_2}$in eq. (20), we get

$$\Delta\int_{t_1}^{t_2} L dt = -\Delta\int_{t_1}^{t_2} H dt \;\; or\; \Delta\int_{t_1}^{t_2}[H + L]dt = 0$$

or $$\Delta\int_{t_1}^{t_2}\textstyle\sum_k p_k\dot{q}_k\, dt = 0 \qquad [\because H = \sum_k p_k\dot{q}_k - L] \quad (22)$$

This is what is known as ***principle of least action.***

The quantity $\int_{t_1}^{t_2}\sum_k p_k\dot{q}_k\, dt$= W is called ***Hamilton's characteristic function.*** Hence the principle of least action can be stated as

$$\Delta W = \Delta\int_{t_1}^{t_2}\textstyle\sum_k p_k\dot{q}_k\, dt = 0 \quad (23)$$

6.5 SUMMARY:

Hamilton's principle is used to derive the equation of motion for a simple harmonic oscillator. By expressing the system's Lagrangian (kinetic minus potential energy) and applying the principle of least action, the familiar equation of motion for sinusoidal oscillation is obtained. This demonstrates the power of Hamilton's principle in solving fundamental problems in mechanics. Hamilton's principle is applied to analyze the motion of a simple pendulum. The Δ-variation method is used to determine the path that minimizes the action integral, leading to the pendulum's equation of motion. This illustrates how variational calculus is utilized within Hamilton's principle to solve for the dynamics of constrained systems. This is the core concept underlying Hamilton's principle. It states that the actual path taken by a physical system between two points in time is the one that minimizes the action, which is the integral of the Lagrangian over time. [1] It is a fundamental variational principle that governs the dynamics of classical systems, and is the basis for the derivation of Lagrange's equations, and Hamilton's equations.

6.6 TECHNICAL TERMS:

Simple applications of Hamilton principle- linear harmonic oscillator, simple pendulum, Δ-variation and principle of Least Action.

6.7 SELF-ASSESSMENT QUESTIONS:

1) Show linear harmonic oscillator as a simple application of Hamilton principle.
2) Derive simple pendulum, Δ-variation.
3) Write the Principle of Least Action.

6.8 SUGGESTED READINGS:

1) Classical Mechanics: H. Goldstein
2) Mechanics: Simon
3) Mechanics: Gupta, Kumar and Sharma

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LESSON-7

CANONICAL TRANSFORMATIONS

7.0 AIM AND OBJECTIVES:

To learn about-

- Equations of canonical transformation (Generating functions)
- Examples of canonical transformations for a harmonic oscillator.

To understand how to transform a set of canonical coordinates (position and momentum) into a new set while preserving the Hamiltonian structure of the system. To develop the ability to find transformations that simplify the equations of motion or reveal hidden symmetries. To learn about the concept of generating functions, which provide a systematic way to define and implement canonical transformations. To define and understand the conditions for a transformation to be canonical. To introduce the four types of generating functions (F1, F2, F3, F4) and their respective roles in defining canonical transformations. To derive the equations that relates the old and new canonical coordinates using the generating functions. To learn how to choose appropriate generating functions to achieve desired transformations. To be able to prove that a transformation is canonical. To understand the importance of canonical transformations in Hamiltonian mechanics. To illustrate the application of canonical transformations in a concrete and solvable system, the harmonic oscillator. To demonstrate how canonical transformations can simplify the analysis of the harmonic oscillator and reveal its underlying structure. To show the practical use of generating functions. To find specific canonical transformations that transform the harmonic oscillator's Hamiltonian into a simpler form (e.g., a constant).To use generating functions to derive the equations for these transformations. To analyze the physical interpretation of the transformed coordinates and momenta. To understand how canonical transformations can be used to solve the harmonic oscillator's equations of motion. To see how a well chosen canonical transformation can greatly simplify a problem. To reinforce the understanding of canonical transformation through a practical example.

STRUCTURE:

7.1 CANONICAL TRANSFORMATIONS:

In several problems, we may need to change one set of position and momentum coordinates into another set of position and momentum coordinates. Suppose that q_k and $p_k$are the old position and momentum coordinates and Q_k and P_k are the new ones. Let these coordinates be related by the following transformations:

$$\left.\begin{array}{l} P_k = P_k(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, t) \\ Q_k = Q_k(p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_n, t) \end{array}\right\} \qquad (1)$$

Now, if there exists a Hamiltonian $\acute{H}$ in the new coordinates such that

$$\dot{P}_k = -\frac{\partial H'}{\partial Q_k} \text{ and } \dot{Q}_k = \frac{\partial H'}{\partial P_k} \qquad (2)$$

where $H' = \sum_{k=1}^{n} P_k \dot{Q}_k - L'$ (3)

and L' substituted in the Hamilton's principle

$$\delta \int L' dt = 0 \qquad (4)$$

Gives the correct equations of motion in terms of the new coordinates P_k and Q_k, then the transformations (1) are known as *canonical (or contact) transformations.*

7.1.1 Generating Functions:

For canonical transformations, the Lagrangian L in p_k, q_k coordinates and L' in P_k, Q_k coordinates must satisfy the Hamilton's principle, i.e.,

$$\delta \int_{t_1}^{t_2} L\, dt = 0 \text{ and } \delta \int_{t_1}^{t_2} L'\, dt = 0 \qquad (5)$$

But $L = \sum_{k=1}^{n} p_k \dot{q}_k - H$ and $L' = \int_{k=1}^{n} P_k \dot{Q}_k - H'$,

Therefore, $\delta \int_{t_1}^{t_2} [\sum_k p_k \dot{q}_k - H]\, dt = 0$ (6)

and $\delta \int_{t_1}^{t_2} [\sum_k p_k \dot{q}_k - H']\, dt = 0$ (7)

subtracting eq. (7) from eq. (6), we get

$$\delta \int_{t_1}^{t_2} [(\sum_k p_k \dot{q}_k - H) - (\sum_k p_k \dot{q}_k - H')]\, dt = 0 \qquad (8)$$

In $\delta \int f dt = 0$ is to be satisfied, in general, by $f = df/dt$, where F is an arbitrary function. Therefore,

$$\delta \int_{t_1}^{t_2} \frac{dF}{dt} dt = 0 \qquad (9)$$

where $\frac{dF}{dt} = L - L'$ (10a)

or $\frac{dF}{dt} = (\sum_k p_k \dot{q}_k - H) - (P_k \dot{Q}_k - H')$ (10 b)

The function F is known as the generating function. The meaning of the name will be clear later on. The first bracket in (10) is a function of p_k, q_k and t and the second as a function P_k, Q_k and $t. F$ is therefore, in general, a function of $(4n + 1)$ variables are subjected to the transformation equation (1) and therefore F may be regarded as the function of $(2n + 1)$ variables, comprising t and any $2n$ of the p_k, q_k, P_k, Q_k. Thus we see that F can be written as a function of $(2n + 1)$ independent variables in the following four forms:

(i) $F_1(q_k, Q_k, t), (ii) F_2\ (q_k, P_k, t) (iii) F_3(p_k, Q_k, t)$ and $(iv) F_4(p_k, P_k, t)$ (17)

The choice of the functional form of the generating function F depends on the problem under consideration.

Case I: if we choose the form *(i), i.e.,*

$$F_1 = F_1(q_1, q_2, \dots, q_k, \dots q_n, Q_1, Q_2, \dots Q_k, \dots, Q_n, t) \tag{18}$$

Then $\frac{dF_1}{dt} = \sum_k \frac{\partial F_1}{\partial q_k}\dot{q}_k + \sum_k \frac{\partial F_1}{\partial Q_k}\dot{Q}_k + \frac{\partial F_1}{\partial t}$ (19)

Subtracting (19) from (16 b), we can write

$$\sum_k \left(p_k - \frac{\partial F_1}{\partial q_k}\right)\dot{q}_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k}\right)\dot{Q}_k + H' - H - \frac{\partial F_1}{\partial t} = 0$$

or

$$\sum_k \left(p_k - \frac{\partial F_1}{\partial q_k}\right)dq_k - \sum_k \left(P_k + \frac{\partial F_1}{\partial Q_k}\right)dQ_k + [H' - H - \frac{\partial F_1}{\partial t}]dt = 0$$

As q_k, Q_k and t may be regarded as independent variables, (20)

$$p_k = \frac{\partial}{\partial q_k}F_1(q_k, Q_k, t), P_k = -\frac{\partial}{\partial Q_k}F_1(q_k, Q_k, t)$$

and $$H' - H = \frac{\partial}{\partial t}F_1(q_k, Q_k, t) \tag{21}$$

In principle, first equation of (21) may be solved to give

$$Q_K = Q_k(q_k, p_k, t) \tag{22}$$

Substituting this in the second equation of (21), one gets

$$P_k = P_K(q_k, p_k, t) \tag{23}$$

In fact, these are the transformation equations (1). Thus we find that transformation equations can be derived from knowledge of the function F. This is why F is known as the **generating function of the transformation.**

Case II: If the generating function is of the type $F_2(q_k,\ P_k, t)$, then it can be dealt with by affecting a Legendre transformation (7):

$$f' - ux, \text{ where } u = \frac{\partial f}{\partial x}$$

Here, since $P_K = -\frac{\partial F_1}{\partial Q_K}$, we have $u = -P_k, x = Q_k, f' = F2 \; and \; f = F_1$

Therefore, $$F_2(q_k, P_k, t) = F_1(q_k, Q_K, t) + \sum_k P_K Q_K \tag{24}$$

Evidently, F_2 is independent of Q_k variables, because

$\frac{\partial F_2}{\partial Q_k} = \frac{\partial F_1}{\partial Q_K} + P_k = -P_k + P_k = 0$ as $\frac{\partial F_1}{\partial Q_K} = -P_k$ in (21)

Using eq. (16)

$$\left(\sum_k p_k \dot{q}_k - H\right) - \left(\sum_k P_k \dot{Q}_k - H'\right) = \frac{d}{dt}[F_2 - \sum_k P_k Q\_K]$$

or $$\frac{dF_2}{dt} = \sum_k p_k \dot{q}_k + \sum_k Q_k \dot{P}_k + H' - H \tag{25}$$

Total time derivative of $F_2(q_k, P_k, t)$ is

$$\frac{dF_2}{dt} = \sum_k \frac{\partial F_2}{\partial q_k} \dot{q}_k + \sum_k \frac{\partial F_2}{\partial P_k} \dot{P}_k + \frac{\partial F_2}{\partial t} \tag{26}$$

From (25) and (26), we get

$$p_k = \frac{\partial F_2}{\partial q_k}, \; Q_k = \frac{\partial F_2}{\partial P_k} and \; H' - H = \frac{\partial F_2}{\partial t} \tag{27}$$

If we look (21) and (27), we find $\frac{\partial F_1}{\partial t} = \frac{\partial F_2}{\partial t}$. Futher as $\frac{\partial F_1}{\partial q_k} = \frac{\partial F_2}{\partial q_k}$, first equation of (21) and that of (27) are identical. Second equation of (27) appears to be different from the second equation of (21), but in fact it is a rearrangement of it.

Case III: We can again relate the third type of generating function $(F_3(p_k, Q_k, t)$ to by a Legendre transformation in view of the relation $p_k = \frac{\partial F_1}{\partial q_k}$. Hence $u = p_k, x = q_k, f' = F_3$ and $f = F_1$. Therefore,

$$F_3(p_k, Q_k, t) = F_1(q_k, Q_k, t) - \sum_k p_k q_k \tag{28}$$

or$F_1(q_k, Q_k, t) = F_3(p_k, Q_k, t) + \sum_k p_k q_k$

Using eq. (16), we have

$$\left(\sum_k p_k \dot{q}_k - H\right) - \left(\sum_k P_k \dot{Q}_k - H'\right) = \frac{dF_1}{dt} = \frac{d}{dt}(F_3 + \sum p_k q_k)$$

or $$\frac{dF_3}{dt} = -\sum_k \dot{p}_k q_k - \sum_k P_k \dot{Q}_k + H' - H$$

Also, $$\frac{dF_3}{dt} = \sum_k \frac{\partial F_3}{\partial p_k} \dot{p}_k + \sum_k \frac{\partial F_3}{\partial Q_k} \dot{Q}_k + \frac{\partial F_3}{\partial t}$$

$$q_k = -\frac{\partial F_3}{\partial p_k}, P_k = -\frac{\partial F_3}{\partial Q_k} \; and \; H' - H = \frac{\partial F_3}{\partial t} \tag{29}$$

Case IV: Using Legendre transformations, the generating function $F_4(p_k, P_k, t)$ can be connected to $F_1(q_k, Q_k, t)$ as

$$F_4(p_k, P_k, t) = F_1(q_k, Q_k, t) + \sum_k P_k Q_k - \sum_k p_k q_k \qquad (30)$$

Using eq. (16), we have

$$\left(\sum_k p_k \dot{q}_k - H\right) - \left(\sum_k P_k \dot{Q}_k - H'\right) = \frac{d}{dt}\Big(F_4 - \sum_k p_k q_k\Big)$$

or$\frac{dF_4}{dt} = -\sum_k q_k \dot{p}_k + \sum_k Q_k \dot{P}_k + H' - H$

But$\frac{dF_4}{dt} = \sum_k \frac{\partial F_4}{\partial p_k}\dot{p}_k + \sum_k \frac{\partial F_4}{\partial P_k}\dot{P}_k + \frac{\partial F_4}{\partial P_k}P_k \stackrel{\cdot}{+} \frac{\partial F_4}{\partial t}$

A comparison of the above two equations gives the fourth set of transformation equations:

$$q_k = -\frac{\partial F_4}{\partial p_k}, Q_k = \frac{\partial F_4}{\partial P_k}, H' - H = \frac{\partial F_4}{\partial t} \qquad (31)$$

examples of canonical transformations for a harmonic oscillator.

7.2 EXAMPLES OF CANONICAL TRANSFORMATIONS FOR A HARMONIC OSCILLATOR:

7.2.1 Procedure for application of canonical transformations:

We note that the relation between H and H' in all the cases has the same from

i.e., $H' = H + \frac{\partial F}{\partial t}$. Now, if F has no explicit time dependence, then $\frac{\partial F}{\partial t} = 0$ and hence

$$H' = H \qquad (32)$$

Thus, when the generating function has no explicit time dependence, the new Hamiltonian H' is obtained from the old Hamiltonian H by substituting for p_k, q_k in terms of the new variables P_k, Q_k. Further we note that the t has been treated as an invariant parameter of the motion and we have not made any provision for a transformation of the time coordinate along with the other coordinates.

If in the new set of coordinates (P_k, Q_k, t) all coordinates Q_k are cyclic, then

$$\dot{P}_k = -\frac{\partial H\prime}{\partial Q_k} = 0 \text{ or } P_k\text{=constant, say } \alpha_k \qquad (33)$$

If the generating function F does not depend on time t explicitly and H is a constant of motion, not depending on time, then from (32) H' is also constant of motion. Thus H' will not involve Q_k and t (explicit time dependence).

Therefore,

$$H(q_k, p_k) = H'(Q_k, P_k) = H'(P_k) = H'(\alpha_1, \alpha_2, ..\alpha_n)$$

Hamilton's equations for Q_k are

$$\dot{Q}_k = \frac{\partial H'}{\partial P_k} = \frac{\partial H'}{\partial \alpha_k} = \omega_k \tag{34}$$

where ω_k's are functions of the α_k's only and are constant in time.

Eq. (34) has the solution

$$Q_k = \omega_k t + \beta_k \tag{35}$$

where β_k's are the constants of integration, determined by the initial conditions.

7.2.2 Conditions for Canonical Transformations:

Suppose $F = F(q_k, Q_k)$ then obviously $\frac{\partial F}{\partial t} = 0$ and $H = H'$ [from (21)].

Further from (21), we have

$P_k = \frac{\partial F}{\partial q_k}$ and $P_k = -\frac{\partial F}{\partial Q_k}$

Also $$dF = \sum_k \frac{\partial F}{\partial q_k} dq_k + \sum_k \frac{\partial F}{\partial Q_k} dQ_k$$

or $$dF = \sum_k p_k \, dq_k - \sum_k P_k \, dQ_k \tag{36}$$

The left hand side of eq. (36) is an exact differential, hence for a given transformation to be canonical, the right hand side of eq. (36) *i.e.*,$\sum_k p_k \, dq_k - \sum_k P_k dQ_k$ must be an exact differential.

7.3 SUMMARY:

Canonical transformations are a fundamental concept in classical mechanics, facilitating the transition between different Hamiltonian formulations. They preserve the form of Hamilton's equations, enabling simplified analysis of dynamical systems. For a transformation to be canonical, it must satisfy specific conditions, such as maintaining the simple structure of phase space and ensuring that the new coordinates and momenta are related through generating functions. These transformations are essential for solving complex problems and understanding the underlying symmetries in mechanical systems. This involves learning how to change a system's position and momentum coordinates while preserving its Hamiltonian structure. Generating functions (F1, F2, F3, F4) are key tools for defining these transformations. The aim is to simplify problem analysis and uncover system symmetries.

This applies the concept to a practical example. The goal is to find transformations that simplify the harmonic oscillator's Hamiltonian, demonstrating how generating functions are used to derive these transformations and make the problem easier to solve. This reinforces the understanding of canonical transformations through a concrete application.

7.4 TECHNICAL TERMS:

Canonical Transformations, Generating Functions.

7.5 SELF-ASSESSMENT QUESTIONS:

1) Define canonical transformations and obtain the transformation equations corresponding to all possible generating functions
2) Discuss the canonical transformations in detail and explain the condition for a transformation to be canonical
3) What are canonical transformation equations? Discuss how transformation equations can be obtained from generating functions F_1 and F_2.
4) What are canonical transformation equations? Obtain canonical transformation equations corresponding to first two types of generating functions.
5) State the condition for canonical transformation and show that the transformation $q = \sqrt{2P}\sin Q \ \ and \ p = \sqrt{2P}\cos Q$ is canonical
6) What are canonical transformations
7) What is the condition for a transformation to be canonical
8) Show that the transformation $P = \frac{1}{2}(P^2 + q^2) and \ Q = \tan^{-1}(\frac{q}{p})$ is canonical
9) Show that the transformation Q = p and P =-q is canonical

7.6 SUGGESTED READINGS:

1) Classical Mechanics by H.Goldstein.
2) Fundamentals of Classical Mechanics by J.C. Upadhyaya.
3) Classical Mechanics by G. Aruldhas, PHI Publishers.
4) The Theory of relativity and applications, Allen Rea.

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LESSON-8

POISSON’S BRACKET

8.0 AIM AND OBJECTIVES: To learn about

- Introduction to Poisson’s bracket notation
- Equations of motion in Poisson bracket
- The fundamentals and the angular momentum in poisson notation.
- Jacobi identity

To introduce and develop a comprehensive understanding of the Poisson bracket formalism as a powerful tool for describing and analyzing classical mechanics. Understand and apply Poisson's bracket notation: Define the Poisson bracket of two functions of canonical variables. Calculate Poisson brackets for given functions. Recognize and utilize the fundamental Poisson bracket relations. Derive and interpret equations of motion using Poisson brackets: Express Hamilton's equations of motion in Poisson bracket form. Understand how the time evolution of a dynamical variable is determined by its Poisson bracket with the Hamiltonian. Solve problems that involve finding the time evolution of variables. Express and analyze angular momentum using Poisson bracket notation: Represent the components of angular momentum in terms of canonical variables. Calculate the Poisson brackets between angular momentum components. Understand the implications of these Poisson bracket relations for the conservation and properties of angular momentum. State and apply the Jacobi identity: State the Jacobi identity for Poisson brackets. Verify the Jacobi identity for given functions. Understand the importance of the Jacobi identity in the consistency and structure of Hamiltonian mechanics.

STRUCTURE:

8.1 INTRODUCTION:

This chapter delves into the concept of Poisson brackets, Equations of motion in Poisson bracket and fundamentals of Poisson bracket notation. Poisson brackets allow for systematically treating dynamical systems and their evolution in phase space. Key topics include the definition of Poisson brackets, their properties, the fundamental Poisson bracket and the Jacobi identity by Poisson notation.

8.2 POISSON'S BRACKET INTRODUCTION:

We know in the case of infinitesimal contact transformations, the changes in the conjugate variables p_k and q_k are given by

$$\delta q_k = \epsilon \frac{\partial G}{\partial p_k} \text{ and } \delta p_k = -\epsilon \frac{\partial G}{\partial q_k} \tag{1}$$

Where $\in$ is an infinitesimal parameter and the generating function $G(q_k, p_k)$ is arbitrary. Now let us consider some function $F(q_k, p_k)$ with the changes δq_k and δp_k in the coordinates q_k and p_k respectively can be expressed as

$$\delta F = \sum_k \left(\frac{\partial F}{\partial q_k} \delta q_k + \frac{\partial F}{\partial p_k} \delta p_k \right) \tag{2}$$

If the transformation (1), generated by the function G, is applied, we get

$$\delta F = \sum_k \left[\frac{\partial F}{\partial q_k} \left(\epsilon \frac{\partial G}{\partial p_k} \right) + \frac{\partial F}{\partial p_k} \left(-\epsilon \frac{\partial G}{\partial q_k} \right) \right]$$

Since the parameter ϵ is independent of q_k and p_k, we have

$$\delta F = \epsilon \left[\sum_k \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) \right] \tag{3}$$

The quantity in the big bracket in (3) is called the **Poisson bracket** of two functions or dynamical variables $F(q_k, p_k)$ and $G(q_k, p_k)$ and is denoted by $[F, G]$. This definition of Poisson bracket is true for F and G, being functions of time, Thus

$$\delta F = \epsilon [F, G] \tag{4}$$

8.3 POISSON'S BRACKETS:

If the functions F and G are defined as

$$[F, G]_{q,p} = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial q_k} \frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k} \frac{\partial G}{\partial q_k} \right) \tag{5}$$

For brevity, we may drop the subscripts q, p and write the Poisson bracket as $[F, G]$. The total time derivative of the function F can be written as

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \sum_{k=1}^{n} \left(\frac{\partial F}{\partial q_k} \dot{q}_k + \frac{\partial F}{\partial p_k} \dot{p}_k \right) \tag{6}$$

Using, Hamilton's equations $\dot{q}_k = \frac{\partial H}{\partial p_k}$ and $-p_k = \frac{\partial H}{\partial q_k}$, eq. (6) is obtained to be

$$\frac{dF}{dt} = \dot{F} = \frac{\partial F}{\partial t} + \sum_{k=1}^{n}\left(\frac{\partial F}{\partial q_k}\frac{\partial H}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial H}{\partial q_k}\right) \quad (7)$$

In view of the definition of Poisson's bracket given by eq. (5), We obtain

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + [F, H] \quad (8)$$

From this equation we see that the function F is a constant of motion, if

$$\frac{dF}{dt} = 0 \; or \; \frac{\partial F}{\partial t} + [F, H] = 0 \quad (9)$$

Now, if the function F does not depend on time explicitly, $\frac{\partial F}{\partial t} = 0$ and then the condition for F to be constant of motion is obtained to be

$$[F, H] = 0 \quad (10)$$

Thus *if a function F does not depend on time explicitly and is a constant of motion, its Poisson bracket with the Hamiltonian vanishes.* In other words, *a function whose Poisson bracket with Hamiltonian vanishes is a constant of motion of motion.* This result does not depend whether H itself is constant of motion.

Equations of motion in Poisson bracket form: Special cases of (8) are

(1) $F = q_k \dot{q}_k = [q_k, H]$ (8a)

(2) $F = p_k, \dot{p}_k = [p_k, H]$ (8b)

(3) $F = H, \dot{H} = \frac{\partial H}{\partial t}$ (8c)

These equations (8a, 8b, 8c) are identical to Hamilton's equations and referred as **equations of motion in Poisson bracket form.**

Properties of Poisson brackets and Fundamental Poisson brackets: The Poisson bracket has the property of antisymmetric, given by

$$[F, G] = -[G, F] \quad (11)$$

Because $[F, G] = \sum_k [\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}] = -\sum_k \left[\frac{\partial G}{\partial q_k}\frac{\partial F}{\partial p_k} - \frac{\partial G}{\partial p_k}\frac{\partial F}{\partial q_k}\right] = -[G, F]$ (11a)

Thus Poisson bracket does not obey the commutative law of algebra. As an application of the Poisson brackets, we are giving below some of the special cases:

(1) When $G = q_1$,

$$[F, q_1] = \sum_k \left[\frac{\partial F}{\partial q_k}\frac{\partial q_1}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial q_1}{\partial q_k}\right] = -\sum_k \frac{\partial F}{\partial p_k}\delta_{lk}$$

Or $[F, q_1] = -\frac{\partial F}{\partial p_l}$ (12)

Also if $F = q_k$, $[q_k, q_l] = -\frac{\partial q_k}{\partial p_l} = 0$ (13)

And if $F = p_k$, $[p_k, q_l] = -\frac{\partial p_k}{\partial p_l} = -\delta_{kl}$ (14)

(2) When $G = p_l [F, p_l] = \sum_k \frac{\partial F}{\partial q_k} \delta_{kl}$

Or $[F, p_l] = \frac{\partial F}{\partial q_l}$ (15)

For $F = p_k, [p_k, p_l] = \frac{\partial p_k}{\partial q_l} = 0$ (16)

And for $F = q_k, [q_k, p_l] = \frac{\partial q_k}{\partial q_l} = \delta_{kl}$ (17)

The above results can be summarized as follows:

$$[q_k, q_l] = [p_k, p_l] = 0 \quad (18)$$

and $$[q_k, p_l] = \delta_{kl} \quad (19)$$

where δ_{kl} is the kronecker delta symbol with the property

$\delta_{kl} = 0$ for $k \neq 1$ and $\delta_{kl} = 1$ for $k = l$

Equations (18) and (19) are called the fundamental Poisson's brackets.

Further from the definition of Poisson bracket of any two dynamical variables or functions, one can obtain the following identities:

$(i)[F, F] = 0$ (20)

$(ii)[F, C] = 0$, C=constant (21)

$(iii)[CF, G] = C[F, G]$ (22)

$(iv)[F_1 + F_2, G] = [F_1, G] + [F_2, G]$ (23)

$(v)[F, G_1 G_2] = G_1[F, G_2] + [F, G_1]G_2$ (24)

$(vi)\frac{\partial}{\partial t}[F, G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right]$ (25)

$(vii)\big[F, [G, K]\big] + \big[G, [K, F]\big] + \big[K, [F, G]\big] = 0$ (Jacobi's identity) (26)

8.4 FUNDAMENTALS OF POISSON BRACKET NOTATION:

The Poisson bracket has the property of antisymmetry, given by

$$[F, G] = -[G, F] \quad (27)$$

Because $[F, G] = \sum_k [\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}] = -\sum_k \left[\frac{\partial G}{\partial q_k}\frac{\partial F}{\partial p_k} - \frac{\partial G}{\partial p_k}\frac{\partial F}{\partial q_k}\right] = -[G, F]$ (28)

Thus Poisson bracket does not obey the commutative law of algebra. As an application of the Poisson brackets, we are giving below some of the special cases:

(1) When $G = q_1$,

$$[F, q_1] = \sum_k \left[\frac{\partial F}{\partial q_k}\frac{\partial q_1}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial q_1}{\partial q_k}\right] = -\sum_k \frac{\partial F}{\partial p_k}\delta_{lk}$$

Or $[F, q_1] = -\frac{\partial F}{\partial p_l}$ (29)

Also if $F = q_k$, $[q_k, q_l] = -\frac{\partial q_k}{\partial p_l} = 0$ (30)

And if $F = p_k$, $[p_k, q_l] = -\frac{\partial p_k}{\partial p_l} = -\delta_{kl}$ (31)

(2) When $G = p_l [F, p_l] = \sum_k \frac{\partial F}{\partial q_k}\delta_{kl}$

Or $[F, p_l] = \frac{\partial F}{\partial q_l}$ (32)

For $F = p_k, [p_k, p_l] = \frac{\partial p_k}{\partial q_l} = 0$ (33)

And for $F = q_k, [q_k, p_l] = \frac{\partial q_k}{\partial q_l} = \delta_{kl}$ (34)

The above results can be summarized as follows:

$[q_k, q_l] = [p_k, p_l] = 0$ (35)

and $[q_k, p_l] = \delta_{kl}$ (36)

where δ_{kl} is the kronecker delta symbol with the property

$\delta_{kl} = 0$ for $k \neq 1$ and $\delta_{kl} = 1$ for $k = l$

Equations (35) and (36) are called the fundamental Poisson's brackets.

Further from the definition of Poisson bracket of any two dynamical variables or functions, one can obtain the following identities:

$(i)[F, F] = 0$ (37)

$(ii)[F, C] = 0$, C=constant (38)

$(iii)[CF, G] = C[F, G]$ (39)

$(iv)[F_1 + F_2, G] = [F_1, G] + [F_2, G]$ (40)

$(v)[F, G_1 G_2] = G_1[F, G_2] + [F, G_1]G_2]$ (41)

$(vi)\frac{\partial}{\partial t}[F, G] = \left[\frac{\partial F}{\partial t}, G\right] + \left[F, \frac{\partial G}{\partial t}\right]$ (42)

$(vii)\big[F, [G, K]\big] + \big[G, [K, F]\big] + \big[K, [F, G]\big] = 0$ (Jacobi's identity) (43)

8.5 ANGULAR MOMENTUM AND POISSON BRACKETS:

Using the definition of linear and angular momentum, a number of interesting and useful Poisson bracket relations can be obtained.

Poisson brackets relations between the components of p and J: According to the definition of angular momentum,

$$J = r \, x \, p = (x\hat{\imath} + y\hat{\jmath} + z\hat{k}) x (p_x\hat{\imath} + p_y\hat{\jmath} + p_z\hat{k})$$

Or $J = (yp_z - zp_y)\hat{\imath} + (zp_x - xp_z)\hat{\jmath} + (xp_y - yp_x)\hat{k}$

Therefore, $J_x = (yp_z - zp_y), J_y = (zp_x - xp_z)$ and $J_z = (xp_y - yp_x)$

From the definition of Poisson bracket (31)

$$[F, G] = \sum_{k=1}^{n} \left(\frac{\partial F}{\partial q_k}\frac{\partial G}{\partial p_k} - \frac{\partial F}{\partial p_k}\frac{\partial G}{\partial q_k}\right)$$

We have,

$$[p_x, p_y] = [p_y, p_z] = [p_z, p_x] = [p_x, p_x] = 0 \qquad (44)$$

N Next, using the result $[F, p_l] = \frac{\partial F}{\partial q_1}$, we have

$$[J_x, p_y] = p_z, \ [J_x, p_z] = -p_y, [J_x, p_x] = 0 \qquad (45\ a)$$

Similarly, $$[J_y, p_x] = -p_z, [J_y, p_y] = 0, [J_y, p_z] = p_x] \qquad (45\ b)$$

$$[J_z, p_x] = p_y, [J_z, p_y] = -p_x, [J_z, p_z] = 0 \qquad (45\ c)$$

Further

$$[J_x, J_y] = \sum_k \left[\frac{\partial J_x}{\partial q_k}\frac{\partial J_y}{\partial p_k} - \frac{\partial J_x}{\partial p_k}\frac{\partial J_y}{\partial q_k}\right]$$

For $q_1 = x, q_2 = y, q_3 = z \, and \, p_1 = p_x, p_2 = p_y, p_3 = p_z,$

$$[J_x, J_y] = \frac{\partial J_x}{\partial x}\frac{\partial J_y}{\partial p_x} - \frac{\partial J_x}{\partial p_x}\frac{\partial J_y}{\partial x} + \frac{\partial J_x}{\partial y}\frac{\partial J_y}{\partial p_y} - \frac{\partial J_x}{\partial p_y}\frac{\partial J_y}{\partial x} + \frac{\partial J_x}{\partial z}\frac{\partial J_y}{\partial p_z} - \frac{\partial J_x}{\partial p_z}\frac{\partial J_y}{\partial z}$$

$$= 0 - 0 + 0 - 0 + (-p_y)(-x) - (y)(p_x)$$

$$= xp_y - pp_x = J_z \qquad (46)$$

Similarly one can prove that

$$[J_y, J_z] = J_x, [J_z, J_x] = J_y \qquad (47)$$

8.6 JACOBI IDENTITY:

If we make a canonical transformation from the old set of variables $(q_k, p_k)$to a new set of variables (Q_k, P_k), then the new equations of motion are,

$$\dot{P}_k = -\frac{\partial H'}{\partial Q_k} \text{ and } \dot{Q}_k = \frac{\partial H'}{\partial P_k} \tag{48}$$

Now, if we require that the transformed Hamiltonian H' is identically zero i.e., $H' = 0$, then equations of motion (1) assume the from,

$\dot{P}_k = 0$ and $\dot{Q}_k = 0$

$P_k = conastant$ and $Q_k = constant$ (49)

Thus the new coordinates and momenta are constants in time and they are cyclic.

Thus new Hamiltonian H' is related to the old Hamiltonian H by the relation

$$H' = H + \frac{\partial F}{\partial t}$$

Which will be zero only when F satisfies the relation

$$H(q_k, p_k, t) + \frac{\partial F}{\partial t} = 0 \tag{50}$$

Where $H(q_k, p_k, t)$ is written for $H(q_1, q_2, \dots q_n, p_1, p_2, \dots p_n, t)$.

For convenience, We take the generating function F as a function of the old coordinates q_k, the new constant momenta P_k and time t i.e., $F_2(q_k, P_k, t)$. Then

$$p_k = \frac{\partial F_2}{\partial q_k} \tag{51}$$

Therefore, $$H\left(q_k, \frac{\partial F_2}{\partial q_k}, t\right) + \frac{\partial F_2}{\partial t} = 0 \tag{52}$$

Let us see what is the physical meaning of the generating function $F_2(q_k, P_k, t)$. The total time derivative of F_2 is

$$\frac{\partial F_2}{\partial t} = \sum_{k=1}^{n} \frac{\partial F_2}{\partial q_k}\dot{q}_k + \sum_{k=1}^{n} \dot{P}_k + \frac{\partial F_2}{\partial t}$$

Here, $\dot{P}_k = 0$, $\frac{\partial F_2}{\partial t} = -H$ from (31) and $\frac{\partial F_2}{\partial q_k} = p_k$ from (30).

Therefore, $$\frac{\partial F_2}{\partial t} = \sum_{k=1}^{n} p_k \dot{q}_k - H = L$$

or $$F_2 = \int L\, dt = S \tag{53}$$

where S is the familiar **action** of the system, known as the **Hamilton's principal function** in relation to the variational principle. Writing $F_2 = S$ in eq. (52), we get

$$H\left(q_k, \frac{\partial S}{\partial q_k}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{54}$$

This is known as ***Hamilton-Jacobi equation*** which is a partial differential equation of first order in $(n+1)$ variables $q_1, q_2, \dots, q_n, t$.

Let the complete solution of equation eq. (54) be of the form

$$S = S\,(q_1, q_2, \dots., q_n, \alpha_1, \alpha_2, \dots \alpha_n, t). \tag{55}$$

where $\alpha_1, \alpha_2, \dots., \alpha_n$ are n independent constants of integration. Here, we have omitted one arbitrary additive constant which has no importance in a generating function because only partial derivatives of the generating function appear in the transformation equations.

In eq. (55), the solution S is a function n coordinates q_k, time t and n independent constants. We can take these n constants of integration as the new constant momenta i.e.,

$$P_k = \alpha_k \tag{56}$$

Now, the n transformation equations

$$p_k = \frac{\partial S\,(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t)}{\partial q_k} \tag{57}$$

These are n equations, which $t = t_0$ (initially) give the n values of α_k in terms of the initial values of q_k and p_k. The other n transformation equations are

$$Q_k = \frac{\partial S}{\partial P_k} = \text{constant, say } \beta_k$$

or

$$\beta_k = \frac{\partial S\,(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t)}{\partial \alpha_k} \tag{58}$$

Similarly one can calculate the constants β_k by using initial conditions i.e., at $t = t_0$, the known initial values of q_k, in eq. (58). Thus $\alpha_k$andβ_k constants are known and eq. (58) will give q_k in terms of α_k, β_k and t i.e.,

$$q_k = q_k(\alpha_1, \alpha_2, \dots., \alpha_n, \beta_1, \beta_2, \dots., \beta_n, t) \tag{59}$$

After performing the differentiation in eq. (58), eq. (59) may be substituted for q_k to obtain momenta p_k will be obtained as functions of constants α_k, β_k and time t i.e.,

$$p_k = p_k(\alpha_1, \alpha_2, \dots \alpha_n, \beta_1, \beta_2, \dots \beta_n, t) \tag{60}$$

In this way we obtain the desired complete solution of the mechanical problem.

Thus we see that the Hamilton's principal function S is the generator of a canonical transformation to constant coordinates (β_k) and momenta (α_k). Also in solving the Hamilton-Jacobi equation, we obtain simultaneously a solution to the mechanical problem.

8.7 SUMMARY:

This lesson explores the Poisson bracket formalism, a powerful tool in classical mechanics that provides an alternative way to express and analyze the equations of motion. Introduction to Poisson's Bracket Notation: This notation offers a compact and elegant way to express relationships between dynamical variables. Equations of Motion in Poisson Bracket: This illustrates the central role of the Hamiltonian in determining the system's dynamics. Fundamentals and Angular Momentum in Poisson Notation: Angular momentum components

(L_x, L_y, L_z) can be expressed and manipulated using Poisson brackets, revealing their relationships and conservation properties. Jacobi Identity: The Jacobi identity is a fundamental property of Poisson brackets. This identity ensures consistency and plays a crucial role in the mathematical structure of classical mechanics, particularly in the context of canonical transformations and symmetries. The Jacobi identity is essential for proving that Poisson brackets are invariant under canonical transformations.

8.8 TECHNICAL TERMS:

Poisson's bracket notation, Equations of motion in Poisson bracket, angular momentum in Poisson notation and Jacobi identity.

8.9 SELF-ASSESSMENT QUESTIONS:

1) Write briefly a note on Poisson's bracket notation.
2) What are equations of motion in Poisson bracket?
3) Explain the fundamentals and the angular momentum in Poisson notation.
4) Write Jacobi identity.

8.10 SUGGESTED READINGS:

1) Classical Mechanics: H. Goldstein.
2) Mechanics: Simon.
3) Mechanics: Gupta, Kumar and Sharma.

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LESSON-9

HAMILTON-JACOBI METHOD

9.0 AIM AND OBJECTIVES:

To learn about

- Introduction
- Hamilton-Jacobi Equation of Hamilton's principal function
- Harmonic oscillator problem as an example of Hamilton-Jacobi method

To introduce students to the Hamilton-Jacobi formulation of classical mechanics and demonstrate its application in solving dynamical problems. Explain the motivation for the Hamilton-Jacobi formulation as an alternative to Lagrangian and Hamiltonian mechanics. Recognize the connection between the Hamilton-Jacobi equation and the concept of a generating function in canonical transformations. Derive and state the Hamilton-Jacobi equation using Hamilton's principal function (S). Define Hamilton's principal function and its relationship to the classical action. Explain the role of the principle of least action in the derivation of the Hamilton-Jacobi equation. Identify the physical meaning of the partial derivatives of Hamilton's principal function. Apply the Hamilton-Jacobi method to solve the one-dimensional harmonic oscillator problem. Determine Hamilton's principal function for the harmonic oscillator. Extract the time-dependent position and momentum of the oscillator from the solution of the Hamilton-Jacobi equation. Compare the Hamilton-Jacobi solution with the solution obtained using other methods (e.g., direct solution of Newton's equations or Hamiltonian mechanics). Appreciate the elegance and power of the Hamilton-Jacobi method in solving certain dynamical problems.

STRUCTURE:

9.1 INTRODUCTION:

The Hamilton-Jacobi equation is a fundamental concept in classical mechanics that reformulates Newtonian mechanics into a more analytical framework. This chapter explores the Hamilton-Jacobi method, emphasizing its application to the harmonic oscillator problem. The method simplifies the process of solving complex dynamical systems by transforming them into a single first-order partial differential equation. By employing this technique, we gain deeper insights into the behavior of systems governed by conservative forces, leading to a clearer understanding of action and phase space. The harmonic oscillator serves as an exemplary case, showcasing the elegance and utility of the Hamilton-Jacobi approach.

9.2 HAMILTON-JACOBI EQUATION OF HAMILTON'S PRINCIPAL FUNCTION:

If we make a canonical transformation from the old set of variables $(q_k, p_k)$to anew set of variables (Q_k, P_k), then the new equations of motion are,

$$\dot{P}_k = -\frac{\partial H'}{\partial Q_k} \text{ and } \dot{Q}_k = \frac{\partial H'}{\partial P_k} \qquad (1)$$

Now, if we require that the transformed Hamiltonian H' is identically zero i.e., $H' = 0$, then equations of motion (1) assume the from,

$\dot{P}_k = 0$ and $\dot{Q}_k = 0$

$P_k = conastant$ and $Q_k = constant$ (2)

Thus the new coordinates and momenta are constants in time and they are cyclic.

Thus new Hamiltonian H' is related to the old Hamiltonian H by the relation

$$H' = H + \frac{\partial F}{\partial t}$$

Which will be zero only when F satisfies the relation

$$H(q_k, p_k, t) + \frac{\partial F}{\partial t} = 0 \qquad (3)$$

Where $H(q_k, p_k, t)$ is written for $H(q_1, q_2, \dots q_n, p_1, p_2, \dots p_n, t)$.

For convenience, We take the generating function F as a function of the old coordinates q_k, the new constant momenta P_k and time t i.e., $F_2(q_k, P_k, t)$. Then

$$p_k = \frac{\partial F_2}{\partial q_k} \qquad (4)$$

Therefore, $$H\left(q_k, \frac{\partial F_2}{\partial q_k}, t\right) + \frac{\partial F_2}{\partial t} = 0 \qquad (5)$$

Let us see what is the physical meaning of the generating function $F_2(q_k, P_k, t)$. The total time derivative of F_2 is

$$\frac{\partial F_2}{\partial t} = \sum_{k=1}^{n} \frac{\partial F_2}{\partial q_k}\dot{q}_k + \sum_{k=1}^{n} \dot{P}_k + \frac{\partial F_2}{\partial t}$$

Here, $\dot{P}_k = 0$, $\frac{\partial F_2}{\partial t} = -H$ from (5) and $\frac{\partial F_2}{\partial q_k} = p_k$ from (4).

Therefore, $$\frac{\partial F_2}{\partial t} = \sum_{k=1}^{n} p_k \dot{q}_k - H = L$$

or $$F_2 = \int L\, dt = S \qquad (6)$$

Where S is the familiar **action** of the system, known as the **Hamilton's principal function** in relation to the variational principle. Writing $F_2 = S$ in eq. (5), we get

$$H\left(q_k, \frac{\partial S}{\partial q_k}, t\right) + \frac{\partial S}{\partial t} = 0 \qquad (7)$$

This is known as ***Hamilton-Jacobi equation*** which is a partial differential equation of first order in $(n + 1)$ variables $q_1, q_2, \dots, q_n, t$.

Let the complete solution of equation eq. (7) be of the form

$$S = S\,(q_1, q_2, \dots, q_n, \alpha_1, \alpha_2, \dots \alpha_n, t). \qquad (8)$$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are n independent constants of integration. Here, we have omitted one arbitrary additive constant which has no importance in a generating function because only partial derivatives of the generating function appear in the transformation equations.

In eq. (8), the solution S is a function n coordinates q_k, time t and n independent constants. We can take these n constants of integration as the new constant momenta i.e.,

$$P_k = \alpha_k \qquad (9)$$

Now, the n transformation equations

$$p_k = \frac{\partial S\,(q_1, \dots, q_n, \alpha_1, \dots, \alpha_n, t)}{\partial q_k} \qquad (10)$$

These are n equations, which $t = t_0$ (initially) give the n values of α_k in terms of the initial values of q_k and p_k. The other n transformation equations are

$$Q_k = \frac{\partial S}{\partial P_k} = \text{constant, say } \beta_k$$

or $$\beta_k = \frac{\partial S\,(q_1,\dots,q_n,\alpha_1,\dots,\alpha_n,t)}{\partial \alpha_k} \qquad (11)$$

Similarly one can calculate the constants β_k by using initial conditions i.e., at $t = t_0$, the known initial values of q_k, in eq. (11). Thus $\alpha_k$andβ_k constants are known and eq. (11) will give q_k in terms of α_k, β_k and t i.e.,

$$q_k = q_k(\alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_n, t) \qquad (12)$$

After performing the differentiation in eq. (11), eq. (12) may be substituted for q_k to obtain momenta p_k will be obtained as functions of constants α_k, β_k and time t i.e.,

$$p_k = p_k(\alpha_1, \alpha_2, \dots \alpha_n, \beta_1, \beta_2, \dots \beta_n, t) \qquad (13)$$

In this way we obtain the desired complete solution of the mechanical problem.

Thus we see that the Hamilton's principal function S is the generator of a canonical transformation to constant coordinates (β_k) and momenta (α_k). Also in solving the Hamilton-Jacobi equation, we obtain simultaneously a solution to the mechanical problem.

9.3 HARMONIC OSCILLATOR PROBLEM AS AN EXAMPLE OF HAMILTON-JACOBI METHOD:

Let us consider a one-dimensional harmonic oscillator. The force acting on the oscillator at a displacement q is

$$F = -kq$$

Where k is force constant.

Potential energy, $V = \int_0^q kq\,dq = \frac{1}{2}kq^2$

Kinetic energy, $T = \frac{1}{2}mv^2 = \frac{p^2}{2m}$

Hamiltonian, $H = T + V$ (conservative system)

or $H = \frac{p^2}{2m} + \frac{1}{2}kq^2$

But $p = \frac{\partial S}{\partial q}$, therefore

$$H = \frac{1}{2}\left[\frac{\partial S}{\partial q}\right]^2 + \frac{1}{2}kq^2 \qquad (14)$$

Hence the Hamilton-Jacobi equation corresponding to this Hamiltonian is

$$\frac{1}{2m}\left[\frac{\partial S}{\partial q}\right]^2 + \frac{1}{2}kq^2 + \frac{\partial S}{\partial t} = 0 \qquad (15)$$

As the explicit dependence of S on t is involved only in the last term of left hand side of eq. (15), a solution to this equation can be assumed in the form

$$S = S_1(q) + S_2(t) \tag{16}$$

Thus
$$\frac{1}{2m}\left[\frac{\partial S_1}{\partial q}\right]^2 + \frac{1}{2}kq^2 = -\frac{\partial S_2}{\partial t} \tag{17}$$

Setting each side of eq. (17) equal to a constant, say α, we get

$$\frac{1}{2m}\left[\frac{\partial S_1}{\partial q}\right]^2 + \frac{1}{2}kq^2 = \alpha, \text{ and } -\frac{\partial S_2}{\partial t} = \alpha$$

So that $\frac{\partial S_1}{\partial q} = \sqrt{2m\,(\alpha - \frac{1}{2}kq^2)}$ and $-\frac{\partial S_2}{\partial t} = \alpha$

Integrating, we get

$S_1 = \int \sqrt{2m\left(\alpha - \frac{1}{2}kq^2\right)dq + C_1}$ and $S_2 = \alpha t + C_2$

Therefore, $S = \int \sqrt{2m\left(\alpha - \frac{1}{2}kq^2\right)dq - \alpha t + C}$

where $C = (C_1 + C_2)$ the constant of integration. It is to be noted that C an additive constant and will not affect the transformation, because to obtain the new position coordinate $\left(Q = \frac{\partial S}{\partial P}\ or\ \beta = \frac{\partial S}{\partial \alpha}\right)$only partialderivative of S with respect to $\alpha(= P$, new momentum) is required. This is why this additive constant C has no effect on transformation and is dropped. Thus

$$S = \int \sqrt{2m\left(\alpha - \frac{1}{2}kq^2\right)dq - \alpha t} \tag{18}$$

We designate the constant α as the new momentum P. The new constant coordinate $(Q = \beta)$ is obtained by the transformation

$$\beta = \frac{\partial S}{\partial \alpha} = \frac{\sqrt{2m}}{2}\int \frac{dq}{\sqrt{\alpha - \frac{1}{2}kq^2}} - t = \sqrt{\frac{m}{2\alpha}}\int \frac{dq}{\sqrt{1 - \frac{kq^2}{2\alpha}}} - t$$

or
$$\beta = \sqrt{\frac{m}{k}}\sin^{-1} q\sqrt{\frac{k}{2\alpha}} - t$$

Therefore, $\sqrt{\frac{m}{k}}\sin^{-1} q\sqrt{\frac{k}{2\alpha}} = t + \beta$ or $\sin^{-1} q\sqrt{\frac{k}{2\alpha}} = \sqrt{\frac{k}{m}}(t + \beta)$

Writing $\omega = \sqrt{k/m}$, we obtain

$$q = \sqrt{\frac{2\alpha}{m\omega^2}}\, sin\, \omega(t + \beta) \tag{19}$$

Which is the familiar solution of the harmonic oscillator.

9.4 SUMMARY:

- This lesson explores the Hamilton-Jacobi equation, a powerful tool in classical mechanics, providing an alternative approach to solving problems compared to Newtonian or Lagrangian/Hamiltonian mechanics.

- **Introduction:**

  - The lesson begins by introducing the concept of the Hamilton-Jacobi equation.
  - It likely highlights the equation's significance in transforming a mechanics problem into a partial differential equation.
  - It may also touch upon the connection between classical and quantum mechanics, as the Hamilton-Jacobi equation provides a bridge to the Schrödinger equation.

- **Hamilton-Jacobi Equation of Hamilton's Principal Function:**

  - This section delves into the derivation and form of the Hamilton-Jacobi equation.
  - It focuses on Hamilton's principal function, denoted as S(q,t), which depends on the generalized coordinates (q) and time (t).
  - The Hamilton-Jacobi equation is a first-order partial differential equation

- The lesson likely explains how solving this equation yields the principal function S, from which the system's motion can be determined.

- It should explain how the partial derivatives of S relate to the momentum. This section demonstrates the application of the Hamilton-Jacobi method using the familiar harmonic oscillator problem. It showcases how to set up the Hamilton-Jacobi equation for the harmonic oscillator. It then guides through the process of solving the partial differential equation to find Hamilton's principal function. Finally, it will demonstrate how the principal function is used to find the equations of motion for the harmonic oscillator. This example highlights the systematic approach of the Hamilton-Jacobi method and its effectiveness in solving classical mechanics problems.

9.5 TECHNICAL TERMS:

Hamilton-Jacobi Equation, Hamilton's principal function, Harmonic oscillator problem.

9.6 SELF-ASSESSMENT QUESTIONS:

1) State and prove Hamilton-Jacobi Equation of Hamilton's principal function.
2) Show Harmonic oscillator problem as an example of Hamilton-Jacobi method.

9.7 SUGGESTED READINGS:

1) Classical Mechanics: H. Goldstein
2) Mechanics: Simon
3) Mechanics: Gupta, Kumar and Sharma

Dr. S. Balamurali Krishna

LESSON-10

HAMILTON-JACOBI EQUATION

10.0 OBJECTIVES:

To learn about

- Hamilton-Jacobi equation for Hamilton's characteristic function
- Action angle variables

To introduce and explain the Hamilton-Jacobi equation as a powerful tool for solving classical mechanics problems, and to develop an understanding of action-angle variables as a method for simplifying the description of integrable systems.Upon completion of this lesson, students should be able to:Derive and understand the Hamilton-Jacobi equation for Hamilton's characteristic function.Apply the Hamilton-Jacobi equation to solve simple mechanical systems.Define and explain action-angle variables. Understand the significance of action-angle variables in simplifying the analysis of periodic motion. Calculate action-angle variables for specific examples, such as the harmonic oscillator.Recognize the connection between the Hamilton-Jacobi theory and the concept of integrability.

STRUCTURE:

10.1 HAMILTON-JACOBI EQUATION FOR HAMILTON'S CHARACTERISTIC FUNCTION:

We were able to obtain the solution of the Hamilton-Jacobi equation, because S could be separated into two parts: $S_1(q)$and$S_2(t)$, where $S_1(q)$ involves the variable q only $S_2(t)$ the variable t only. In this case, the Hamiltonian H was not involving time explicitly. However, such a separation of variables is always possible, if the Hamiltonian H does not involve time t explicitly. This method is often called the ***method of separation of variables.***

If the Hamiltonian H is not an explicit function of time t, then the Hamiltonian-Jacobi equation for S is obtained to be

$$H\left[q_k, \frac{\partial S}{\partial q_k}\right] + \frac{\partial S}{\partial t} = 0 \qquad (1)$$

Since the first term involves the dependence of S on q_k and the second term on t, we can assume the solution S in the form

$$S(q_k, \alpha_k, t) = W(Q_k, \alpha_k) - \alpha_1 t \qquad (2)$$

Therefore, $\frac{\partial S}{\partial q_k} = \frac{\partial W}{\partial q_k}$ and $\frac{\partial S}{\partial t} = -\alpha_1$

And hence the Hamilton-Jacobi equation assumes the form

$$H\left[q_k, \frac{\partial W}{\partial q_k}\right] = \alpha_1$$

or $$H\left(q_1, q_2, \dots q_n, \frac{\partial W}{\partial q_1}, \frac{\partial W}{\partial q_2}, \dots, \frac{\partial W}{\partial q_n}\right) = \alpha_1 \qquad (3)$$

This is the time-independent Hamilton-Jacobi equation. The constnat of integration α_1 is thus equal to the constant value of H. For conservative system, $H = \alpha_1 = E$, where E represents the total energy of the system. Thus *for conservative system, Hamiltonian-Jacobi equation* is wtitten as

$$H\left[q_k, \frac{\partial W}{\partial q_k}\right] = E \qquad (4)$$

The eq.(3) can also be obtained directly by taking W as the generating function $W(q_k, P_k)$ independent of time. The equations of transformations are

$$p_k = \partial W / \partial q_k \text{ and } Q_k = \partial W / \partial P_k \qquad (5)$$

Now if the new momenta P_k are all constants of motion α_k, where α_1 in particular is the constant of motion H, then $Q_k = \partial W / \partial \alpha_k$. The condition to determine W is that

$$H(q_k, p_k) = \alpha_1$$

Using $p_k = \frac{\partial W}{\partial q_k}$, we obtain

$$H\left[q_k, \frac{\partial W}{\partial q_k}\right] = \alpha_1$$

Which is identical to eq. (3)

Also $$H' = H + \frac{\partial W}{\partial t}$$

But $W(q_k, P_k)$ does not involve time and hence

$$H' = H = \alpha_1 \; (= E \text{ for conservative system}) \tag{6}$$

The function W is known as Hamilton's characterstic function. It generates a canonical transformation where all the new coordinates Q_k are cyclic because $H' = \alpha_1$, depending only on one of the new momenta $P_1 = \alpha_1$ and does not contain any Q_k. Now the canonical equations for new variables are

$$\dot{P}_k = -\frac{\partial H'}{\partial Q_k} = 0 \text{ or } P_k = \alpha_k, \text{ constant} \tag{7}$$

and$\dot{Q}_k = \frac{\partial H'}{\partial \alpha_k} = 1$ for $k = 1$ and $\dot{Q}_k = 0$ for $k \neq 1$.

Hence the solutions are

$$Q_1 = t + \beta_1 = \partial W / \partial \alpha_1 \text{ for } k = 1 \tag{7a}$$

and
$$Q_k = \beta_k = \partial W / \partial \alpha_k \text{ for } k \neq 1 \tag{7b}$$

Thus out of all the new coordinates Q_{Kk}, Q_1 is the only coordinate which is not a constant of motion. Here we observe the conjugate relationship between the time as the new coordinate and Hamiltonian (energy) as the conjugate momentum.

The Hamilton-Jacobi equationdetermines the dependence of the Hamilton's characterstic functon W on the old coordinates q_k. A complete solution of this equation will have n constants of ntegration and as explained earlier and in the discussion of harmonic oscillator problem, one of them is just an additive constant.

Rest of the $n-1$ independent constants $\alpha_1, \alpha_3, \dots, \alpha_n$ plus α_1 may then as new constant momenta. First half of the equations (29), when evaluated with the initial condition $t = 0$, relates the n constant α_k to the initial values of q_k and p_k. Physical significance of the Hamilton's characterstic fnctin W: The function W has physical significance similar to the Hamilton's principal function S. Since $W(q_k, P_k)$does not involve time t explicitly, its total time derivative is

$$\frac{dW}{dt} = \sum_{k=1}^{n} \frac{\partial W}{\partial q_k} \dot{q}_k + \sum_{k=1}^{n} \frac{\partial W}{\partial P_k} \dot{P}_k$$

Since α_k, constants, $\dot{P}_k = 0$ and therefore

$$\frac{dW}{dt} = \sum_{k=1}^{n} p_k \dot{q}_k$$

or
$$W = \int \textstyle\sum_k p_k \dot{q}_k dt = \int \sum_k p_k dq_k \tag{8}$$

*Sometimes it is useful to have a set of n indeoendent functions of α_k as the transformed momenta i.e.,

$$p_k = \gamma_k(\alpha_1, \alpha_2, \dots, \alpha_n)$$

Now, $W = W(q_k, \gamma_k)$ and the Hamiltonian H or H' will, in general, depend on morethan one of the γ_k's. The equations of motion for Q_k are

$$Q_k = \frac{\partial H'}{\partial \gamma_k} = f_k$$

where$f_k{}'$ s are the functions of γ_k.

Therefore, $Q_k = f_k t + \beta_k$

Thus now all the new coordinates are linear functions of time.

which is the abbreviated action,

and $S = \int L\, dt = \int \sum_k [p_k \dot{q}_k - H] dt = W - \int H\, dt$

When H does not involve time t explicitly $\int H\, dt = \alpha_1 t$. so that

$S = W - \alpha_1 t$ or $S(q_k, P_k, t) = W(q_k, P_k) - \alpha_1 t$

or $$S(q_k, t) = W(q_k) - Et \qquad (9)$$

where $P_k = \alpha_k$ are constants and $\alpha_1 E$, total energy.

It is to be remarked that when the Hamiltonian does not involve time explicitly, one can solve a mechanical problem by using either Hamilton's principal function or Hamilton's characterstic function.

10.2 ACTION ANGLE VARIABLES:

We have seen that Hamiltonian Formalism allows much wider class of coordinate transformations. A suitable choice of new conjugate variables can simplify the situation drastically. In many cases there is a natural choice of variables, which simplify the problem - they are known as "action-angle variables".

Consider a more general example of a 1-d system with potential V (q), which has (local) minimum. The trajectories in phase-space will be closed orbits with constant total energy. We conjectured and proved that the correct choice for the actions variable is

$$I(E) = \frac{1}{2\pi} \oint p\, dq,$$

that is the area of phase-space enclosed by an orbit (multiplied by $1/2\pi$). It is a function of E only. The angle variable can be calculated as

$$\theta = \frac{d}{dI}\int pdq.$$

We note that all 1-d systems are integrable because E is conserved. We discussed two ways of proving the above result. First, by direct integration of

$$dt = \sqrt{\frac{m}{2}}\frac{dq}{\sqrt{E - V(q)}}$$

over one orbit. Second, alternative proof uses generating function. Here I describe the latter. Consider F(q, θ). The motion is periodic in q, p, therefore it must be periodic in θ. Thus, F is periodic in θ. Recall the expression for the differential of F

$$dF = pdq - PdQ + (H' - H)dt.$$

In our case H′ = H and we obtain

$$dF = pdq - Id\theta.$$

Let us integrate over a single period: q returns to its original value, while θ changes by amount, which we choose to be period 2π. From periodicity of F it follows that

$$\oint dF = 0 = \oint pdq - \oint Id\theta = \oint pdq - 2\pi I \quad \Rightarrow \quad \theta = \frac{d}{dI}\int pdq.$$

10.3 SUMMARY:

This lesson delves into two advanced topics in classical mechanics: the Hamilton-Jacobi equation and action-angle variables.

- **Hamilton-Jacobi Equation:**
  - We begin by introducing the Hamilton-Jacobi equation, a first-order partial differential equation that can be used to determine the motion of a system.
  - We explore how Hamilton's characteristic function, a solution to this equation, can be used to find the canonical transformation that renders the Hamiltonian time-independent.
  - This approach provides an alternative and often more elegant method for solving classical mechanics problems compared to traditional Newtonian or Lagrangian methods.

- **Action-Angle Variables:**
  - We then transition to action-angle variables, a set of canonical coordinates particularly useful for analyzing systems with periodic or quasi-periodic motion.
  - Action variables are constants of motion, while angle variables increase linearly with time, simplifying the description of the system's evolution.
  - We demonstrate how action angle variables simplify the description of systems such as the harmonic oscillator, and other integrable systems.
  - The connection between these topics is that the Hamilton-Jacobi equation is a tool that can be used to find canonical transformations, which are used to find the action angle variables.
  - We will also discuss how the Hamilton-Jacobi theory connects to the idea of integrability, and how it is used to find the conserved quantities of a system.

10.4 TECHNICAL TERMS:

Hamilton-Jacobi equation, Hamilton's characteristic function and Action angle variables.

10.5 SELF-ASSESSMENT QUESTIONS:

1) Derive Hamilton-Jacobi equation for Hamilton's characteristic function?
2) Write Action angle variables?

10.6 SUGGESTED READINGS:

1) Classical Mechanics: H. Goldstein
2) Mechanics: Simon
3) Mechanics: Gupta, Kumar and Sharma

Dr. S. Balamurali Krishna

LESSON-11

EULER ANGLES

11.0 AIM AND OBJECTIVES:

To learn about

- The Euler angles-first rotation, second rotation and third rotation
- Angular momentum
- Inertia tensor

The aim of this lesson is to provide an understanding of the key concepts related to rotational dynamics, focusing on Euler angles, angular momentum, and the inertia tensor. The lesson will explore how these concepts are used to describe the motion of rigid bodies and how they are interconnected in classical mechanics.

By the end of the lesson, students will be able to:

1) **Euler Angles**: Understand the concept of Euler angles and their use in describing the orientation of a rotating body. Students will learn how to apply these angles to describe the first, second, and third rotations during the movement of a rigid body.
2) **Angular Momentum**: Grasp the concept of angular momentum, how it is calculated, and its importance in understanding the rotational motion of a system. Students will learn how angular momentum relates to torque and conservation laws.
3) **Inertia Tensor**: Understand the inertia tensor and its role in characterizing the rotational inertia of a rigid body. Students will learn how the tensor is used to calculate the moment of inertia for arbitrary axes of rotation.
4) Analyze the interrelationship between these concepts to solve real-world problems related to rigid body rotation.

STRUCTURE:

11.1 EULER'S ANGLES:

We are interested in knowing three independent parameters to specify the orientation of body set of axes relative to the space set of axes. For this purpose, we use three angles. These angles may be chosen in various ways, but the most commonly used set of three angles are the Euler's angles, represented by $\emptyset, \theta$ and ψ.

We can reach an arbitrary orientation of the body set of axes X', Y', Z' from space set of axes (X, Y, Z) by making three successive rotations performed in a specific order.

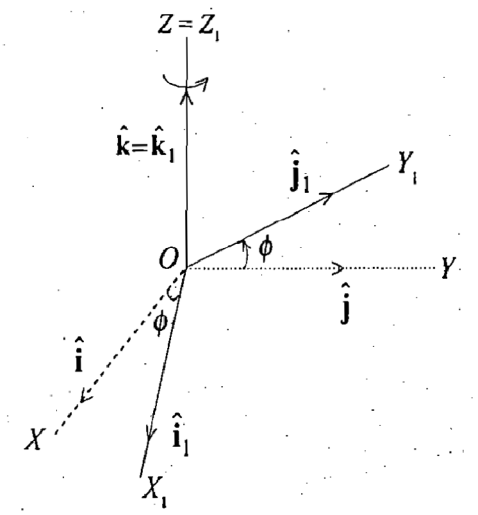

Fig. 11.1: Euler's Angles-First Rotation ϕ, Defining Precession Angle

1) **First rotation (ϕ):** First the space set of axes is rotated through an angle ϕ counter-clock wise about the Z-axis so that Y-Z plane takes the new position $Y_1 - Z_1$ and this new plane $Y_1 - Z_1$ contains the Z' axis of the body coordinate system. Now the new position of the coordinate system $X_1Y_1Z_1$ (with $Z = Z_1$) [Fig. 11.1]. If $\hat{\imath}', \hat{\jmath}', \hat{k}'$ are the unit vectors along X, Y, Z axes and $\hat{\imath}_1, \hat{\jmath}_1, \hat{k}_1$ along $X_1Y_1Z_1$ axes respectively, then the transformation to this new set of axes from space set of axes is represented by the equations

$$\hat{\imath}_1 = cos\phi\hat{\imath} + sin\phi\hat{\jmath}$$

$$\hat{\jmath}_1 = -sin\phi\hat{\imath} + cos\phi\hat{\jmath} \qquad (1)$$

$$\hat{k}_1 = \hat{k}$$

or
$$\begin{bmatrix} \hat{\imath}_1 \\ \hat{\jmath}_1 \\ \hat{k}_1 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} \qquad (2)$$

Thus XYZ axes are transformed $X_1Y_1Z_1$ by the matrix of tranformation

$$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (3)$$

The angle ϕ is called the **precession angle.**

2) **Second Rotation ($\boldsymbol{\theta}$):** Next intermediate axes$X_1Y_1Z_1$are rotated about X_1 axis counter clock wise through an angle θ to the position $X_2Y_2Z_2$ so that Y_1, Z_1 axes acquire the positions Y_2, Z_2 with $Z_2 = Z'$ [Fig.11.2]. This also results the plane $X_2.Y_2$ in plane $X'Y'$. If $\hat{\imath}_2, \hat{\jmath}_2, \hat{k}_2$ are unit vectors along $X_2Y_2Z_2$ axes respectively, then

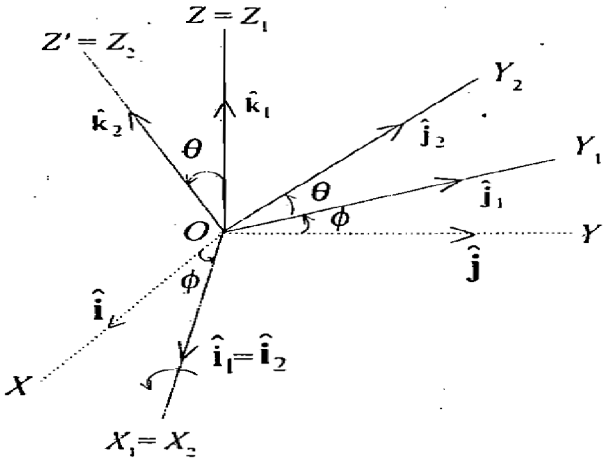

Fig. 11.2: Euler's Angles-Second Rotation $\boldsymbol{\theta}$, defining Nutation Angle

$$\hat{\imath}_1 = (\hat{\imath}_1.\hat{\imath})\hat{\imath} + (\hat{\imath}_1.\hat{\jmath})\hat{\jmath} + (\hat{\imath}_1.\hat{k})\hat{k}$$

$$= cos\phi\hat{\imath} + cos\left(\frac{\pi}{2} - \phi\right)\hat{\jmath} + cos\frac{\pi}{2}\hat{k}$$

$$= cos\phi\hat{\imath} + sin\phi\hat{\jmath}$$

$$\hat{\jmath}_1 = (\hat{\jmath}_1.\hat{\imath})\hat{\imath} + (\hat{\jmath}_1.\hat{\jmath})\hat{\jmath} + (\hat{\jmath}_1.\hat{k})\hat{k}$$

$$= cos\left(\frac{\pi}{2} + \phi\right)\hat{\imath} + cos\phi\hat{\jmath} = -sin\phi\hat{\imath} + cos\phi\hat{\jmath}$$

or
$$\begin{bmatrix} \hat{\imath}_2 \\ \hat{\jmath}_2 \\ \hat{k}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix} \begin{bmatrix} \hat{\imath}_1 \\ \hat{\jmath}_1 \\ \hat{k}_1 \end{bmatrix} \qquad (4)$$

In this case the matrix of transformation is

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & sin\theta \\ 0 & -sin\theta & cos\theta \end{bmatrix} \tag{5}$$

The angle θ is called the nutation angle.The $X_2 = X_1$ axis is at the intersection of the $X - Y$ and $X_2 - Y_2$ planes and is called the line of nodes.

3) **Third rotation (ψ):** Finally the third rotation is performed about $Z_2 = z'$ axis through an angle ψ counter-clockwise so that X_2, Y_2 axes coincide $X_3 = X', Y_3 = Y'$ [Fig. 11.3].

Thus these three rotations $\phi, \theta, and\ \psi$ bring the space set of axes to coincide with body set of axes. Thus $\phi, \theta, and\ \psi$ angles can be taken as three generalized coordinates. Now

$$\hat{\imath}_3 = \hat{\imath}' = \hat{\imath}_2 cos\psi + \hat{\jmath}_2 sin\psi$$

$$\hat{\jmath}_3 = \hat{\jmath}' = -\hat{\imath}_2 sin\psi + \hat{\jmath}_2 cos\psi$$

$$\hat{k}_3 = \hat{k}' = \hat{k}_2$$

$$\begin{bmatrix} \hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}' \end{bmatrix} = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{\imath}_2 \\ \hat{\jmath}_2 \\ \hat{k}_2 \end{bmatrix} \tag{6}$$

So that the transformation matrix is

$$B = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{7}$$

The angle ψ is called the **body angle.**

In this way we have reached at the body set of axes after three successive rotations of space set of axes. We may write the complte matrix of transformation $\boldsymbol{A}$ as

$$\begin{bmatrix} \hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}' \end{bmatrix} = A \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} \ or \ \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{8}$$

But using (2), (3), (4), (5), (6) and (7), we get

$$\begin{bmatrix} \hat{\imath}' \\ \hat{\jmath}' \\ \hat{k}' \end{bmatrix} = \begin{bmatrix} \hat{\imath}_3 \\ \hat{\jmath}_3 \\ \hat{k}_3 \end{bmatrix} = B \begin{bmatrix} \hat{\imath}_2 \\ \hat{\jmath}_2 \\ \hat{k}_2 \end{bmatrix} = BC \begin{bmatrix} \hat{\imath}_1 \\ \hat{\jmath}_1 \\ \hat{k}_1 \end{bmatrix} = BCD \begin{bmatrix} \hat{\imath} \\ \hat{\jmath} \\ \hat{k} \end{bmatrix} \tag{9}$$

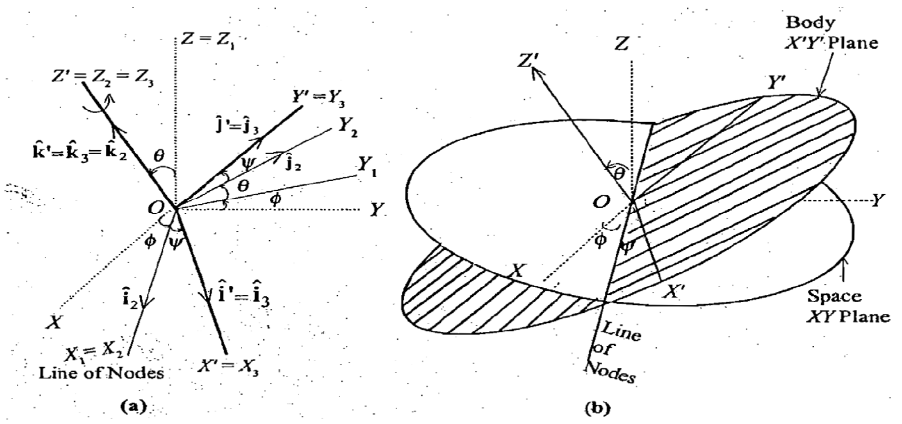

Fig. 11.4: Euler's angles-Third rotation ψ, defining body angle, (b) The three Eulerian angle ϕ, θ and ψ in different planes.

From (8) and (9) we see that the complete matrix of tranformation from space set of axes to body set of axes is

$$A = BCD \qquad (10)$$

The inverse transformation from body set of axes to space set of axes will be given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix}$$

Now

$$A = BCD = \begin{bmatrix} cos\psi & sin\psi & 0 \\ -sin\psi & cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & sin\phi & 0 \\ -cos\theta sin\phi & cos\theta cos\phi & sin\theta \\ sin\theta sin\phi & -sin\theta cos\phi & cos\theta \end{bmatrix} \text{-------} (11)$$

The inverse tranformation matrix from body set of axes to space set of axes is given by $A^{-1} = A_T$because A represents a proper orthogonal matrix. Thus

$$A^{-1} = \begin{bmatrix} cos\psi\, cos\phi - cos\theta\, sin\phi\, sin\psi & -sin\psi\, cos\phi - cos\psi\, cos\theta\, sin\phi & sin\theta\, sin\phi \\ cos\psi sin\phi + sin\psi cos\theta cos\phi & -sin\psi sin\phi + cos\psi cos\theta\, cos\phi & -sin\theta cos\phi \\ sin\psi sin\theta & cos\psi sin\theta & cos\theta \end{bmatrix} \text{----}$$

--(12)

11.2 ANGULAR MOMENTUM AND INERTIA TENSOR:

Considered as a linear operator that transforms into L, the matrix I has elements that behave as the elements of a second-rank tensor. The quantity I is therefore identified as a second-rank tensor and is usually called the moment of inertia tensor or briefly the inertia tensor.

The kinetic energy of motion about a point is

$$T = \frac{1}{2} m_i v_i{}^2, \tag{1}$$

where v, is the velocity of the ith particle relative to the fixed point as measured in the space axes. By Eq. (5.2), T may also be written as

$$\mathrm{T} = \frac{1}{2} m_i v_i \,.\, (\,\omega \times r_i) \tag{2}$$

which, upon permuting the vectors in the triple dot product, becomes

$$\mathrm{T} = \frac{\omega}{2} \,.\, m_i \; (r_i \;\times\; v_i) \tag{3}$$

The quantity summed over i will be recognized as the angular momentum of the body about the origin, and in consequence, the kinetic energy can be written in the form

$$\mathrm{T} = \frac{\omega.L}{2} = \frac{\omega.I.\omega}{2} \tag{4}$$

Let n be a unit vector in the direction of w so that ω = ωπ. Then an alternative form for the kinetic energy is

$$\mathrm{T} = \frac{\omega^2}{2}\; n.I.n = \frac{1}{2} I\omega^2 \tag{5}$$

where I is a scalar, defined by

$$\mathrm{I} = \mathrm{n.\ 1.\ n} = m_l(r^2{}_i - (r_i.n)^2) \tag{6}$$

and known as the moment of inertia about the axis of rotation. In the usual elementary discussions, the moment of inertia about an axis is defined as the sum, over the particles of the body, of the product of the particle mass and the square of the perpendicular distance from the axis. It must be shown that this definition is in accord with the expression given in Eq. (6). The perpendicular distance is equal to the magnitude of the vector r_i x n. Therefore, the customary definition of I may be written as

$$\mathrm{I} = m_i\ (r_i\ \mathrm{xn}).\ (r_i\ \mathrm{x\ n}). \tag{7}$$

Multiplying and dividing by w^2, this definition of I may also be written as

$$\mathrm{I} = \frac{m_i}{\omega^2} (\,\omega \times r_i).(\,\omega \times r_i)$$

But each vector in the dot product is exactly the relative velocity v, as measured in the space system of axes. Hence, I so defined is related to the kinetic energy by

$$\mathrm{I} = \frac{2T}{\omega^2}\,,$$

which is the same as Eq (6), and therefore I must be identical with the scalar defined by Eq. (7). The value of the moment of inertia depends upon the direction of the axis of rotation. As a usually changes its direction with respect to the body in the course of time, the moment of inertia must also be considered a function of time. When the body is constrained so as to rotate only about a fixed axis, then the moment of inertia is a constant. In such a case, the kinetic energy (4) is almost in the form required to fashion the Lagrangian and the equations of motion. The one further step needed is to express was the time derivative of some angle, which can usually be done without difficulty.

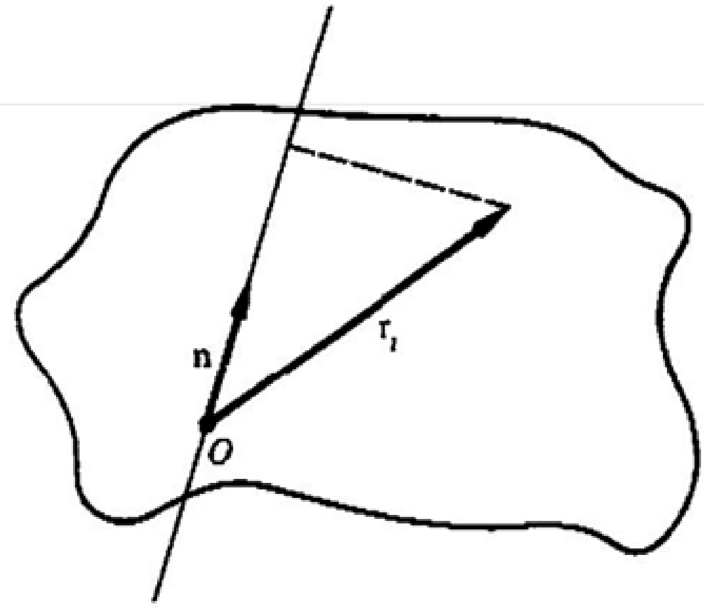

Fig. 11.5: The definition of Moment of Inertia

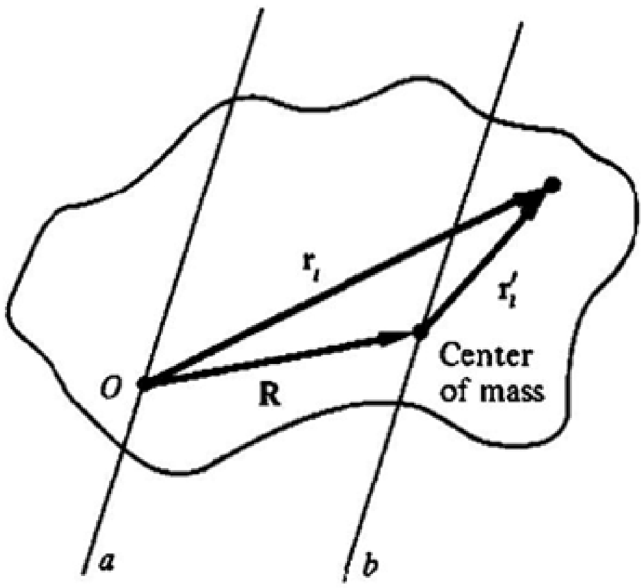

Fig. 11.6: The Vectors involved in the Relation between Moments of Inertia about Parallel Axes

Along with the inertia tensor, the moment of inertia also depends upon the choice of origin of the body set of axes. However, the moment of inertia about some given axis is related simply to the moment about a parallel axis through the center of mass. Let the vector from the given origin O to the center of mass be R, and let the radii vectors from O and the center of mass to the ith particle be r, and r, respectively. The three vectors so defined are connected by the relation (Fig. 3)

$$r_l = R + r'_l. \qquad (8)$$

The moment of inertia about the axis a is therefore

$$I_a = m_i\,(r_i\, xn)^2 = m_i\,[(r'_l. + R)\ x\ n]^2$$

Or

$$I_a = M\,(R\ x\ n)^2 + m_i\,(r'_i\ x\ n)^2\ +2m_i\,(R\ x\ n).\ (r'_i\ x\ n),$$

where M is the total mass of the body. The last term in this expression can be rearranged as

$$-2(R\ x\ n)\ (n\ x\ m_i\,.r'_i\ )$$

By the definition of center of mass, the summation mır vanishes. Hence, la can be expressed in terms of the moment about the parallel axis b as

$I_a = I_b + M(R\ x\ n)^2 =$

$I_b + MR^2 \sin^2\theta.$ (9)

The magnitude of Rxn, which has the value R sine, where is the angle between R and n, is the perpendicular distance of the center of mass from the axis passing through O. Consequently, the moment of inertia about a given axis is equal to the moment of inertia about a parallel axis through the center of mass plus the moment of inertia of the body, as if concentrated at the center of mass, concerning the original axis.

The inertia tensor is defined in general from the kinetic energy of rotation about

an axıs, and is written as

$$T_{rotation} = \frac{1}{2} m_i(\omega\ \times r_i)^2 = \frac{1}{2}\omega_\alpha\omega_\beta m_i\left(\delta_{\alpha\beta}r_i^2 - r_{i\alpha}r_{i\beta}\right)$$

where Greek letters indicate the components of & and r,. In an inertial frame, the sum is over the particles in the body, and ria is the ath component of the position of the ith particle. Because Trotation is a bilinear form in the components of w, it can be written as

$$T_{rotation} = \frac{1}{2} I_{\alpha\beta}\,\omega_\alpha\omega_\beta,$$

Where

$$I_{\alpha\beta} = m_i\left(\delta_{\alpha\beta} r_i^2 - r_{i\alpha} r_{i\beta}\right) \qquad (10)$$

is the moment of inertia tensor. To get the moment of inertia about an axis through the center of mass, choose the rotation about this axis For a body with a contin- uous distribution of density p(r), the sums in the components of the moment of inertia tensor in Eq. (10) reduce to

$$I_{\alpha\beta} = \int \rho(r)\left(r^2 \delta_{\alpha\beta} - r_\alpha r_\beta\right) dV \qquad (11)$$

As an example, let us consider a homogeneous cube of density p, mass M, and side a. Choose the origin to be at one corner and the three edges adjacent to that corner to lie on the +x, +y, and +z axes. If we define b = Ma², then straightforward integration of Eq. (11) gives

$$I = \begin{pmatrix} \frac{2}{3}b & -\frac{1}{4}b & -\frac{1}{4}b \\ -\frac{1}{4}b & \frac{2}{3}b & -\frac{1}{4}b \\ -\frac{1}{4}b & -\frac{1}{4}b & \frac{2}{3}b \end{pmatrix}$$

Thus, both the moment of inertia and the inertia tensor possess a type of revolution, relative to the center of mass, very similar to that found for the linear and angular momentum and the kinetic energy.

11.4 EULER'S EQUATIONS OF MOTION FOR A RIGID BODY:

11.4.1 Newtonian Method: If a rigid body is rotating under the action of a torque τ with one point fixed, then the torque is expressed as

$$\tau = \left[\frac{dJ}{dt}\right]_s \qquad (1)$$

Where J is the angular momentum and its time derivative refers to the space set of axes, represented by the subscript s, because the equation holds in an inertial frame.

The body coordinate system is rotating with an instantaneous angular velocity m. The time derivatives of angular momentum J in the body coordinate and space coordinate systems are related as

$$\left[\frac{dJ}{dt}\right]_s = \left[\frac{dJ}{dt}\right]_b + \omega \times J \qquad (2)$$

Thus

$$\tau = \frac{dJ}{dt} + \omega \times J \qquad (3)$$

where we have dropped the body subscript because we shall represent the physical quantities of right hand side in the body coordinate system.

We choose principal axes for body set of axes. If I_1, $I_2$and I_3 are the principal moments of inertia, then

$$J = I_1\omega_1\hat{\imath} + I_2\omega_2\hat{\jmath} + I_3\omega_3\hat{k} \qquad (4)$$

Where $\omega = \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}$ is the angular velocity with components, ω_1, ω_2 and ω_3 along the principal axis.

As the principal moments of inertia and body base vectors $\hat{\imath}$,$\hat{\jmath}$ and $\hat{k}$ are constants in time with respect to the body coordinate system, we find that in the body coordinate system, using the time derivative of J is

$$\frac{dJ}{dt} = I_1\dot{\omega}_1\hat{\imath} + I_2\dot{\omega}_2\hat{\jmath} + I_3\dot{\omega}_3\hat{k} \qquad (5)$$

Substituting in(3), we obtain

$$\tau = I_1\dot{\omega}_1\hat{\imath} + I_2\dot{\omega}_2\hat{\jmath} + I_3\dot{\omega}_3\hat{k} + \left(\omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}\right) \times (I_1\omega_1\hat{\imath} + I_2\omega_2\hat{\jmath} + I_3\omega_3\hat{k}) \qquad (6)$$

Writing $\tau = \tau_1\hat{\imath} + \tau_2\hat{\jmath} + \tau_3\hat{k}$, we can obtain the *x, y, z* components of the torque τ as

$$\tau_1 = I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 \qquad (7)$$

$$\tau_2 = I_2\dot{\omega}_2 + (I_3 - I_1)\omega_3\omega_1 \qquad (8)$$

$$\tau_3 = I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 \qquad (9)$$

Eqs. (6) are known as Euler's equations for the motion of a rigid body with one point fixed under the action of a torque. These equations can also be derived from Lagrange's equations, when the generalized forces G, are the torques and Euler's angles (ϕ, θ, ψ) are the generalized coordinates.

11.4.2 Lagrange's Method: When a rigid body, is rotating with one point fixed, Euler's angles completely describe the orientation of the rigid body. In case of the rotating rigid body, we take the Euler's angles (ϕ, θ, ψ) as the generalized coordinates and components of the applied torque as the generalized forces corresponding to these angles. For conservative system, Lagrangian for the systems is

$$L = T\left(\dot{\varphi}, \dot{\theta}, \dot{\psi}, \varphi, \theta, \psi\right) - V(\varphi, \theta, \psi) \qquad (1)$$

where T is the rotational kinetic energy and is given by

$$\tau = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \qquad (2)$$

Where the body axes are taken as principal axes.

In view of the angular velocity components ω_1, $\omega_{2,}$ ω_3 along the principal axes can be written as

$$\omega_1 = \dot{\varphi} sin\theta sin\psi + \dot{\theta} cos\psi$$

$$\omega_2 = \dot{\varphi} sin\theta cos\psi + \dot{\theta} cos\psi \qquad (3)$$

$$\omega_3 = \varphi cos\dot{\theta} + \dot{\psi}$$

The Lagrangie's equation for ψ coordinate is

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{\psi}}\right] - \frac{\partial L}{\partial \psi} = 0 \qquad (4)$$

But for L=T-V, given by (2),

$$\frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\psi}}\right] - \frac{\partial T}{\partial \psi} = -\frac{\partial V}{\partial \psi} \qquad (5)$$

Because $\frac{\partial V}{\partial \dot{\psi}}$=0

However, the angle y is the angle of rotation about the principal Z-axis and is one of the generalized coordinates in the present problem. The generalized force $\left[G = -\frac{\partial V}{\partial \psi}\right]$ coresponding to the generalized coordinate ψ is obviously the Z-component of the impressed torque i.e.,

$$\tau_3 = G = -\frac{\partial V}{\partial \psi}$$

$$\tau_3 = \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{\psi}}\right] - \frac{\partial T}{\partial \psi}$$

$$\tau_3 = \frac{d}{dt}\left[\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}}\right] - \frac{\partial}{\partial \psi}\left[\sum_i \frac{\partial T}{\partial \omega_i}\frac{\partial \omega_i}{\partial \dot{\psi}}\right] \qquad (6)$$

But from (65), we get

$$T = \frac{1}{2}\sum_i I_i\omega_i^2$$

Therefore, $\frac{\partial T}{\partial \omega_i} = I_i \omega_i$

From (4), we obtain

$$\frac{\partial \omega_1}{\partial \dot{\psi}} = \frac{\partial \omega_2}{\partial \dot{\psi}} = 0 \, and \, \frac{\partial \omega_3}{\partial \dot{\psi}} = 1$$

So that

$$\sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \dot{\psi}} = I_3 \omega_3 \qquad (7)$$

Also from (4), we get

$$\frac{\partial \omega_1}{\partial \psi} = -\dot{\varphi} sin\theta cos\psi - \dot{\theta} sin\psi = \omega_2$$

$$\frac{\partial \omega_2}{\partial \psi} = -\dot{\varphi} sin\theta sin\psi - \dot{\theta} cos \sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \dot{\psi}} = -\omega_1$$

$$\frac{\partial \omega_3}{\partial \psi} = 0$$

Hence

$$\sum_i \frac{\partial T}{\partial \omega_i} \frac{\partial \omega_i}{\partial \psi} = \frac{\partial T}{\partial \omega_1} \frac{\partial \omega_1}{\partial \psi} + \frac{\partial T}{\partial \omega_2} \frac{\partial \omega_2}{\partial \psi} + \frac{\partial T}{\partial \omega_3} \frac{\partial \omega_3}{\partial \psi}$$

$$= I_1 \omega_1 \omega_2 + I_2 \omega_2 (-\omega_1) = -(I_2 - I_1) \omega_1 \omega_2 \qquad (8)$$

Substituting the values from (2) or (7) and (8) in (2), we get

$$\tau_3 = \frac{d}{dt}(I_3 \omega_3) + (I_2 - I_1) \omega_1 \omega_2$$

or

$$\tau_3 = I_3 \dot{\omega}_3 + (I_2 - I_1) \omega_1 \omega_2 \qquad (9)$$

Which is the third Euler's equation obtained earlier. One may obtain the other two Euler's equations by simply cyclic permutation. Not that these two equations do not correspond to θ and ϕ coordinates.

In case a rigid body is rotating about a fixed axis, say principal Z-axis, then

$$\omega_1 = \omega_2 = 0 \, and \, \omega_3 = \omega$$

Therefore, from eqs. (9) We have the equations of motion as

$$\tau_1 = \tau_2 = 0$$

$$\tau_3 = I_3 \dot{\omega} \; or \; \tau = I\omega \qquad (10)$$

and

Where we have put $\tau_3=\tau$ and $I_3=I$ corresponding to Z-axis is

Instantaneous angular momentum about Z-axis is

$$J_3=I_3\omega_3 \text{ or } J=I\omega \qquad (11)$$

And instantaneous rotational kinetic energy is

$$T = \frac{1}{2}\omega \cdot J = \frac{1}{2}I\omega^2 \qquad (12)$$

11.5 SUMMARY:

This lesson covers the fundamental principles of rotational dynamics, focusing on three key concepts: Euler angles, angular momentum, and the inertia tensor. Euler angles are used to describe the orientation of a rotating body in three-dimensional space through a series of rotations. Angular momentum is introduced as a key quantity that describes a body’s rotational motion and is related to the forces acting on the system. The inertia tensor is a mathematical representation that encodes the distribution of mass in a rigid body and is crucial for determining its rotational behavior. Together, these concepts provide a comprehensive framework for analyzing and understanding the dynamics of rotating systems in classical mechanics.

11.6 TECHNICAL TERMS:

Euler angles, Angular momentum and Inertia tensor.

11.7 SELF-ASSESSMENT QUESTIONS:

1) Write a brief note on Euler angles.
2) What is Angular momentum and Inertia tensor?

11.8 SUGGESTED READINGS:

1) Classical Mechanics: H.Goldstein
2) Mechanics: Simon
3) Mechanics: Gupta, Kumar and Sharma

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LESSON-12

DYNAMICS OF RIGID BODY

12.0 AIM AND OBJECTIVES:

To learn about-

- Principal axes and principal moments of inertia
- Rotational kinetic energy of a rigid body
- Torque-free motion of a rigid body

The aim of this lesson is to explore the concepts of principal axes and moments of inertia, rotational kinetic energy, and torque-free motion in rigid body dynamics. The lesson will focus on understanding how these principles are essential for analyzing the motion of a rigid body and how they contribute to a deeper understanding of rotational motion in classical mechanics.

By the end of the lesson, students will be able to:

1) Principal Axes and Principal Moments of Inertia: Understand the concept of principal axes and principal moments of inertia. Students will learn how to find the principal axes of a rigid body and calculate its corresponding moments of inertia, and why they are crucial in simplifying rotational motion problems.
2) Rotational Kinetic Energy: Grasp the concept of rotational kinetic energy, and how it is related to the moment of inertia and angular velocity of a rigid body. Students will learn how to calculate rotational kinetic energy and understand its significance in energy conservation in rotational systems.
3) Torque-Free Motion of a Rigid Body: Understand the dynamics of a rigid body that is not subject to any external torque, and how it behaves under such conditions. Students will learn the implications of torque-free motion, such as the conservation of angular momentum and the importance of initial conditions in determining the body's motion.

STRUCTURE:

12.1 PRINCIPAL AXES AND PRINCIPAL MOMENTS OF INERTIA:

A point mass moving along a circular path has an angular velocity vector, ω~, directed along the axis of the circle, and an angular momentum vector, L~, relative to the center of the circle which is parallel to the angular velocity. The quantities are related in magnitude b y L = M R2ω where M is the particle mass and R is the radius of the circle. The combination M R2 is the moment of inertia of the point mass relative to the axis of rotation.1 An extended rigid body may be viewed as a distribution of point masses. If such a rigid body rotates about some fixed axis, the angular velocity vector and the angular momentum vector are not, in general, parallel. However, a relation bet ween ω~ and the component of L~ which is parallel to ω~ can still be written down. The proportionality factor is the moment of inertia of the rigid body relative to the axis of rotation,

$$L_\omega = I_\omega \omega, \qquad (1)$$

where L ω is the component of L~ in the same direction as ω~ . That is, L~· ω~ = Lω cos α ≡ L ω ω, α being the angle bet ween L~ and ω~. I is the moment of inertia of the rigid body relative to the axis of rotation determined b y the vector ω~ . For an extended rigid object it is the analog of what M R2 is for a point object of mass M moving in a circle of radius R. In fact, b y considering the rigid body as a collection of point masses, each its own distance from the axis of rotation, one can directly calculate the moment of inertia for the object under consideration.2 The other components of L~ (those perpendicular to ω~) cannot be related to ω~ via a relation such as Eq.(1). Suppose that L~ is parallel to ω~. Then L~ has no components perpendicular to ω~, and Eq.(1) can be written as a vector equation:

L~ = I ω~. (2)

The conditions under which Eq. (2) is satisfied usually exist if the angular velocity ω~ is directed along one of the symmetry axes of the object. 3 These are called the principal axes of inertia of the object and the moments of inertia relative to these axes are the principal moments of inertia. As an illustration of these concepts, consider the following situation. Suppose that relative to a fixed inertial reference frame the angular momentum vector of a rotating rigid body is given by: L~ = [A sin(α)sin(ωt + δ)] xˆ + [A sin(α) cos(ωt + δ)] yˆ + [A cos α]zˆ. Here A, α and δ are constants, t is the time, and ω is the magnitude of the angular velocity of the rotating rigid body. The vector ω~ has magnitude and direction given b y ω~ = ωzˆ. The quantities xˆ, yˆ, and zˆ are unit vectors along the coordinate axes of the inertial reference frame. It is clear that L~ and ω~ are not parallel (ω~ does not have x- and y-components, L~ does). The axis of rotation of this object is thus not one of the principal axes of inertia.

12.2 ROTATIONAL KINETIC ENERGY OF A RIGID BODY:

The rotational kinetic energy of the solid body is

$$Trot = \frac{1}{2}\sum mv^2 = \sum v \cdot mv = \frac{1}{2}\sum(\omega \times r) \quad (29)$$

The triple scalar product is the volume of a parallelepiped, which justifies the next step:

$$\text{Trot}=\frac{1}{2}\sum \omega\cdot(r\times p) \quad (30)$$

All particles have the same angular velocity, so:

$$\text{Trot}=\frac{1}{2}\omega\cdot\sum(r\times p)=\frac{1}{2}\omega\cdot L=\frac{1}{2}\omega\cdot I\ \omega \quad (31)$$

Thus we arrive at the following expressions for the rotational kinetic energy:

$$\text{Trot}=\frac{1}{2}\omega L=\frac{1}{2}\omega I\ \omega \quad (32)$$

If it is rotating about a principal axis, they *are* parallel, and the expression reduces to the familiar $\frac{1}{2}I\ \omega^2$.

12.3 TORQUE-FREE MOTION OF A RIGID BODY:

(1) Equations of Motion: When a rigid body is not subjected to any net toque, the Euler's equations of motion of the body with one point fixed reduced to

$$I\text{-}_1\dot{\omega}_1 = (I_2 - I_3)\omega_2\dot{\omega}_3 \quad (1)$$

$$I\text{-}_2\dot{\omega}_2 = (I_3 - I_1)\omega_3\dot{\omega}_1 \quad (2)$$

$$I\text{-}_3\dot{\omega}_3 = (I_1 - I_2)\omega_1\dot{\omega}_2 \quad (3)$$

In case the body is not subjected to any net forces or torques, its centre of mass is either at rest or moves with uniform velocity. Obviously we may discuss the rotational motion of the rigid body in a reference system in which the center of mass is stationary and choose the centre of mass as fixed point and origin for the principal axes in the body. In such case, we obtain from (12) two integrals of motion, describing the kinetic energy and angular momentum as constant in time.

If we multiply eqs (12) by ω_1, $\omega_{2,}$ ω_3 respectively and then add, we obtain

$$I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3 = (I_2 - I_3 + I_3 - I_1 + I_1 - I_2)\omega_1\omega_2\omega_3 = 0$$

$$\frac{d}{dt}\left(\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2\right) = 0$$

$$\frac{1}{2}I_1\omega_1^2 + \frac{1}{2}I_2\omega_2^2 + \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}\omega.J = constant \quad (4)$$

Which is the principle of **conservation of total rotational kinetic energy** in absence of external torque.

As

$$\tau = \frac{dJ}{dt} = 0$$

$$J = I_1\omega_1\hat{\imath} + I_2\omega_2\hat{\jmath} + I_3\omega_3\hat{k} = constant \qquad (5)$$

(2) Geometric description of the rigid body motion: In case of torque–free motion of rigid body, we have written above eqs. (12) and consequently two integrals of motion (4) and (5). Now we describe an interesting geometrical description of the motion, called as **Pointsot's construction**. In this context, first we shall describe intertia ellipsoid.

(A) Intertia Ellipsoid: The kinetic energy of rotating rigid body in a coordinate system of principal axes is given by

$$T = \frac{1}{2}\omega.J = \frac{1}{2}\sum_{\alpha=1}^{3} I_\alpha\omega_\alpha^2$$

When angular velocity ω is expressed as $\omega = \omega\hat{n} = \omega_1\hat{\imath} + \omega_2\hat{\jmath} + \omega_3\hat{k}$ and I_1, I_2, $I_3$are the principal moments of inertia and I is the moment of inertia about the axis of rotation. Thus we have

$$I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2 = I\omega^2 = 2T \qquad (6)$$

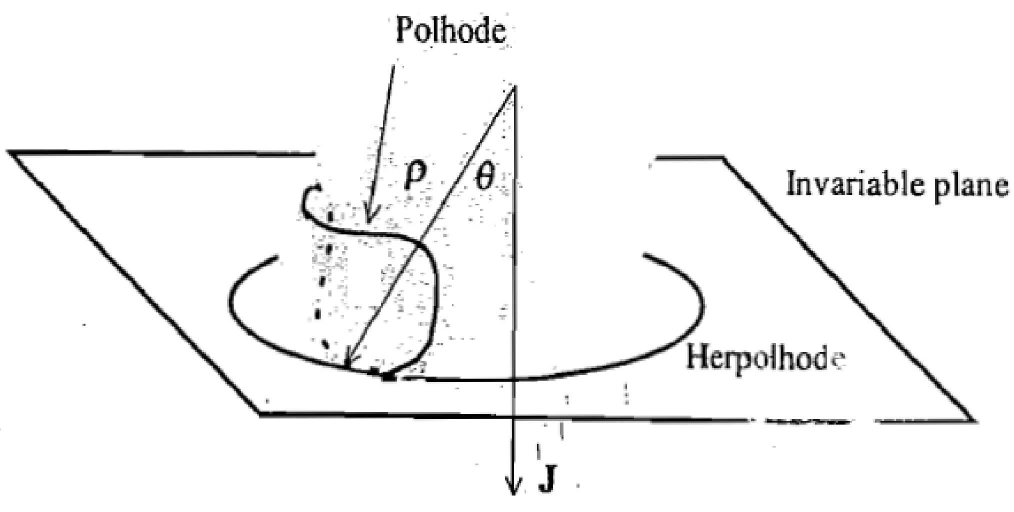

Fig. 12.5: The Motion of the Inertia Ellipsoid Relative to the Invariable Plane

Let us define avector

$\rho = \frac{\hat{n}}{\sqrt{I}} = \frac{\omega}{\omega\sqrt{I}} = \frac{\omega}{\sqrt{I\omega^2}} = \frac{\omega}{\sqrt{2T}}$ (7)

$\rho = \rho_1\hat{\imath} + \rho_2\hat{\jmath} + \rho_3\hat{k}$ (8)

So that $\rho_1 = \frac{\omega_1}{\sqrt{2T}}$ etc. are the components of ρ vector along principal axes.

Hence eq.

$I_1\rho_1^2 + I_2\rho_2^2 + I_3\rho_3^2 = 1$ (9)

This equation represents an ellipsoid in ρ-space which is called as which is called as **inertia ellipsoid** As the direction of the axis of rotation changes in time, the p vector along the same direction moves accordingly and its top moves on the surface of the inertial ellipsoid.

(B) Invariable Plane: Let a rigid body be rotating about a fixed point O. The body is not subjected to any external force or torque. Therefore, in absence of external torque, the angular momentum vector J is constant and has a fixed direction in space from figure. The line along the direction of the angular momentum vector is known as **invariable** line.

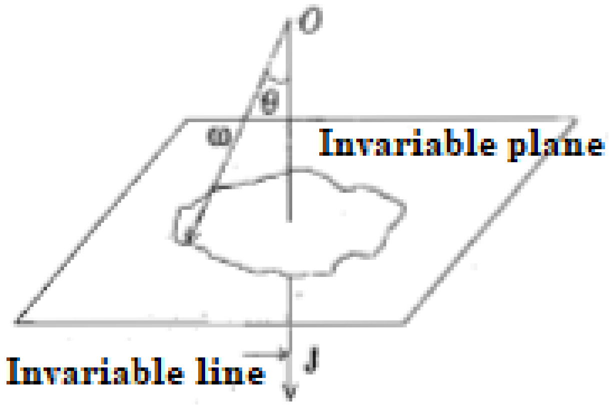

Fig. 12.6: Invariable Line and Plane

For force free motion, the kinetic energy is also constant and hence

$\omega \cdot J = 2T = constant$ (10)

Thus the projection of ω on J (ωcosθ) is constant and therefore the tip of ω describes a plane, called as the **invariable plane**. Now, as the body rotates, an observer, fixed in the body coordinate system, would see a rotation or precession of the angular velocity vector ω with time about the angular momentum vector **J**.

(C) Motion of the inertia ellipsoid on invariable plane : Since from eq. (7), $\rho = \frac{\omega}{\sqrt{2T}}$, this gives for force-free motion

$\rho \cdot J = \frac{\omega \cdot J}{\sqrt{2T}} = \sqrt{2T} = constant$ (8)

because the kinetic energy is the constant of motion. Therefore the tip of ρ also describes an invariable plane in ρ-space. In fact this plane is the tangent plane at the point ρ. Let us prove this statement. From eq. (8)

$$I_1\rho_1^2 + I_2\rho_2^2 + I_3\rho_3^2 = 1 = F(\rho) \text{ (say)} \tag{9}$$

Now,

$$\frac{\partial F}{\partial \rho_1} = 2I_1\rho_1, \frac{\partial F}{\partial \rho_2} = 2I_2\rho_2, \frac{\partial F}{\partial \rho_3} = 2I_3\rho_3 \tag{10}$$

Therefore,

$$\nabla_0 F = 2(I_1\rho_1\hat{\imath} + I_2\rho_2\hat{\jmath} + I_3\rho_3\hat{k}) \tag{11}$$

$$= \frac{2}{\sqrt{2T}}\left(I_1\omega_1\hat{\imath} + I_2\omega_2\hat{\jmath} + I_3\omega_3\hat{k}\right) = \sqrt{\frac{2}{T}}\,J \tag{12}$$

Thus the normal at the point ρ on the ellipsoid (in case force-free motion) is along the constant angular momentum vector J and the tangent plane at the point ρ is perpendicular to J. But the invariable plane is normal to the vector J and hence the tangent plane at ρ is the invariable plane.

The distance between the origin of the ellipsoid and the tangent plane at the point p is given by above Fig.

$$\rho\cos\theta = \frac{\rho\cdot J}{J} = \frac{\omega\cdot J}{J\sqrt{2T}} = \frac{\sqrt{2T}}{J} = constant \tag{13}$$

In the present problem, we find that the distance between the origin of the ellipsoid and invariable plane reinains constant in time. Thus as the angular velocity vector ω and hence ρ changes with time, the inertia ellipsoid rolls (without slipping) on the invariabte plane* with the centre of the ellipsoid at a constant height above the plane. The curve traced on the invariable plane by the point of contact with the ellipsoid is called the herpolhode and the corresponding curve described on the ellipsoid is called the Polhode. In other words, we can say that the polhode rolls without slipping on the herpolhode in the invariable plane. The polhode is a closed curve on the inertia ellipsoid because the inertia ellipsoid would move in order to maintain the heiglit of its origin above the invariable plane. However, the herpolhode, in general, is not a closed curve on the invariable plane.

We have discussed the Poinsot's geometrical construction to describe the force-free motion of a rigid body. The values of kinetic energy T and angular momentum J deterimine the direction of the invariable plane and the height of the centre of the ellipsoid above it.

Hence one may trace out the polhode and the herpolhode. The direction of the angular velocity ω is the same as that of the vector ρ and the instantaneous orientation of the body is given by the orientation of the ellipsoid , which is flxed in the body.

Let us discuss the Poinsot's geometrical discussion for a symmetrical rigid body for which $I_1=I_2$. In this case, the inertia ellipsoid is an ellipsoid of revolution. The ρ vector and hence the angular velocity vector ω will remain constant in magnitude. Consequently the polhode is a circle about the symmetry axis of the ellipsoid and herpolhode is a circle on the invariable plane. An observer sees that the angular velocity vector ω moves on the surface of a cone. This is called **body cone** and its intersection with the inertia ellipsoid is the polhode. An observer, fixed in the space, sees also the angular vetocity vector ω to move on the surface of cone, called as **space cone**. The intersection of this space cone with the invariable plane gives the herpolhode. In this way, **the free tnotion ofa symtnetrical rigid body is described as tlie rolling of body cone on the spaceone**. If $I_3< I_1$, the body cone is outside the space cone and if $I_3>I_1$, the body cone rolls around the inside of the space cone [Fig]. In both cases the two cones are tangent to each other along the instantaneous axis of rotation. In any case, the direction of the angular ve1ocity.vector ω precesses in time about the axis of symmetry of the body.

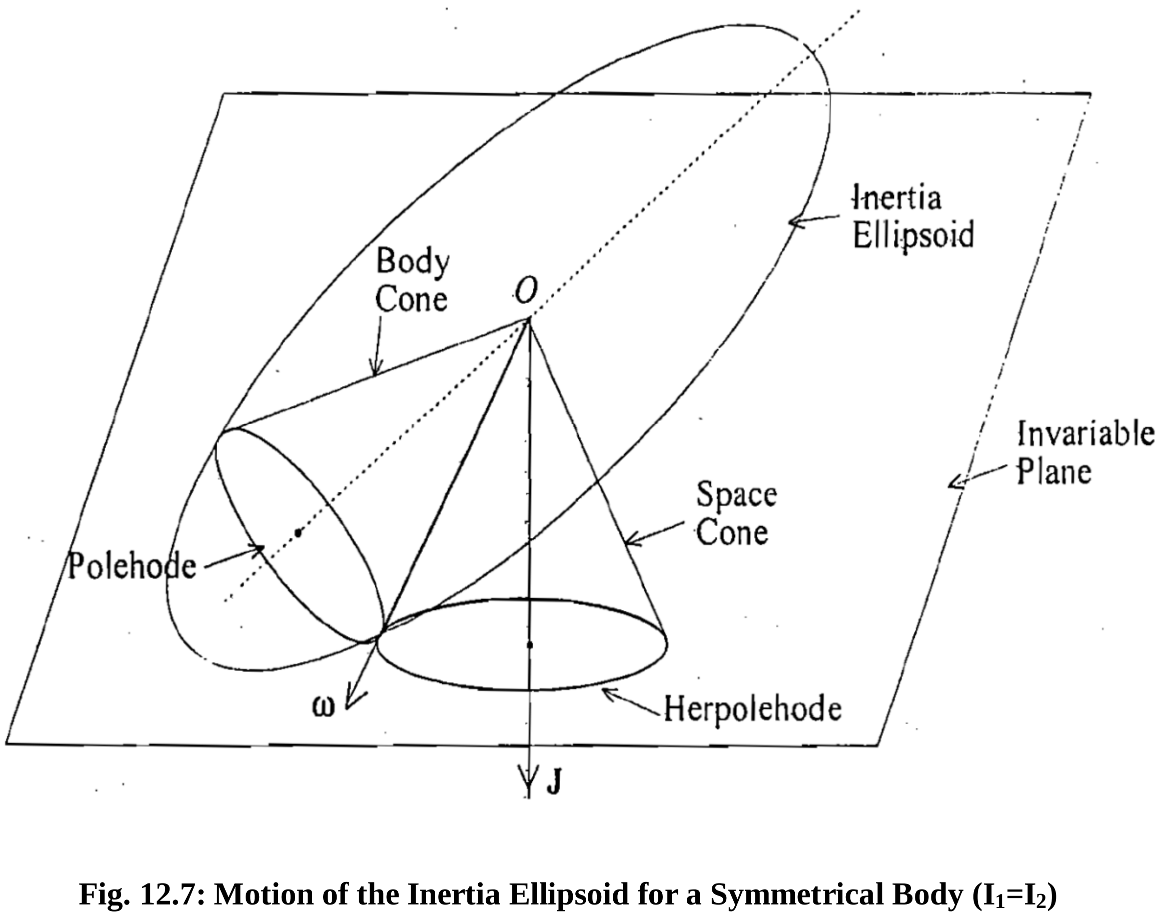

Fig. 12.7: Motion of the Inertia Ellipsoid for a Symmetrical Body ($I_1=I_2$)

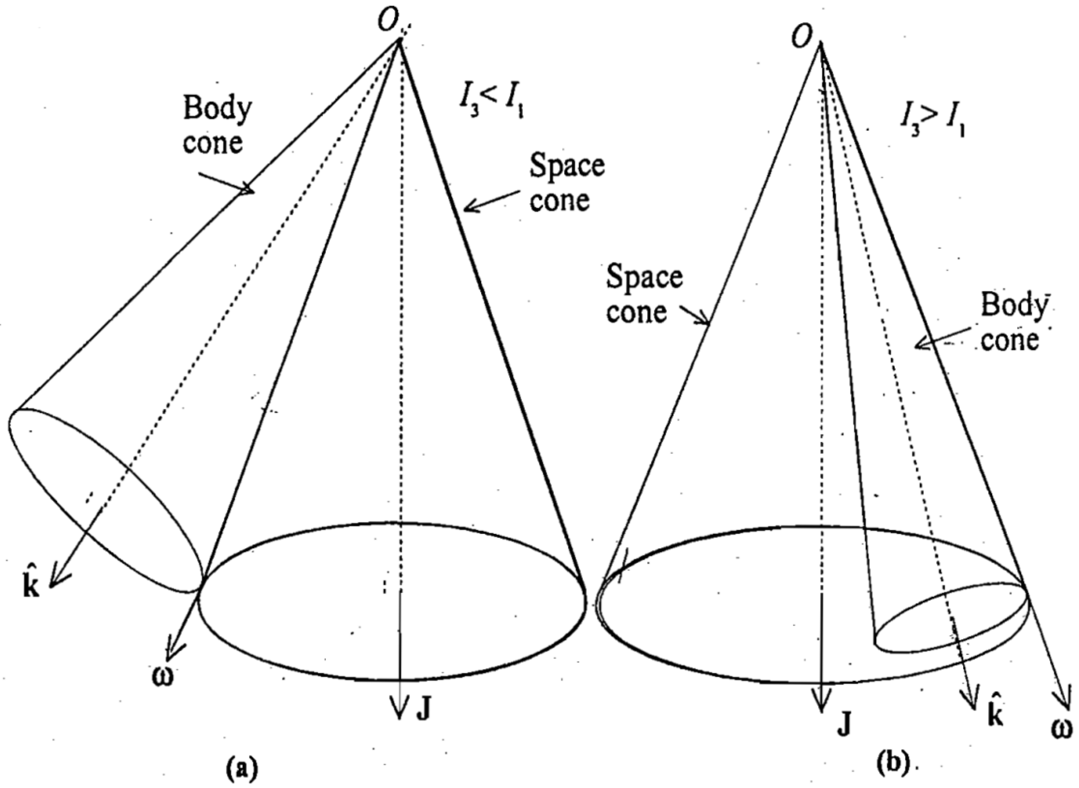

Fig. 12.8: Body Cone Rolling Around a Space Cone without Slipping; (a) outside ($I_3<I_1$)(b) inside ($I_3>I_1$).

Poinsot's geometrical discussion, described above, is in accordance with that obtained by using Euler's equations for a rotating rigid body.

12.4 SUMMARY:

This lesson introduces important concepts related to the rotational motion of rigid bodies. The principal axes and moments of inertia are explored, highlighting how they simplify the analysis of a body's rotational behavior by identifying the axes about which rotation is most stable. The lesson also covers the concept of rotational kinetic energy, emphasizing how it depends on the distribution of mass (via the moment of inertia) and the angular velocity of the body. Finally, the concept of torque-free motion is examined, with a focus on the conservation of angular momentum and its role in predicting the future motion of a rigid body when no external torque is acting. Together, these concepts form a comprehensive foundation for understanding the rotational dynamics of rigid bodies.

12.5 TECHNICAL TERMS:

Moment of Inertia, rigid body and Torque free motion.

12.6 SELF-ASSESSMENT QUESTIONS:

1) Write about the Principal axes and principal moments of inertia.
2) Derive the Rotational kinetic energy of a rigid body.
3) What is Torque-free motion of a rigid body?

12.7 SUGGESTED READINGS:

1) Classical Mechanics: H.Goldstein.
2) Mechanics: Simon.
3) Mechanics: Gupta, Kumar and Sharma.

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LESSON-13

THEORY OF RELATIVITY-I

13.0 AIM AND OBJECTIVES:

To learn about-

- Introduction to special theory of relativity
- Galilean transformations
- principle of relativity

The aim of this lesson is to introduce students to the foundational concepts of the special theory of relativity, including Galilean transformations and the principle of relativity. The lesson will provide a basic understanding of how these concepts revolutionized our view of space, time, and motion, and laid the groundwork for modern physics.

By the end of the lesson, students will be able to:

1) **Introduction to Special Theory of Relativity**: Understand the fundamental concepts behind the special theory of relativity, including the invariance of the speed of light and the relationship between space and time. Students will also explore how Einstein's theory challenges classical Newtonian mechanics.
2) **Galilean Transformations**: Learn about the Galilean transformation equations, which describe the relationship between space and time coordinates in different inertial frames of reference under classical mechanics, and understand how they are limited compared to relativistic transformations.
3) **Principle of Relativity**: Grasp the principle of relativity, which states that the laws of physics are the same in all inertial frames of reference. Students will explore how this principle leads to the idea that the speed of light is constant for all observers, regardless of their motion.

STRUCTURE:

13.1 INTRODUCTION:

The chapter on "Introduction to the Theory of Relativity" lays the groundwork for understanding the transformative concepts that revolutionized classical mechanics. It begins by exploring the Principle of Relativity, which asserts that the laws of physics are the same in all inertial frames. The chapter delves into Galilean Transformations, which describe how physical quantities change between different inertial observers. It discusses the transformation of force and emphasizes the covariance of physical laws. Finally, the relationship between the Principle of Relativity and the constant speed of light is highlighted, paving the way for Einstein's revolutionary insights and their implications for modern physics.

13.2 PRINCIPLE OF RELATIVITY:

Absolute velocity of a body has no meaning. The velocity has a meaning only when it is measured relative to some other body or frame of reference. If two bodies are moving with uniform relative velocity it is impossible to decide which of them is at rest or which of them is moving. This is known as **principle of relativity**. However, acceleration has an absolute meaning. For example, if we are sitting in a windowless accelerated aircraft, we can perform an experiment and measure its acceleration. But if the aircraft is moving with uniform velocity, we cannot measure its velocity. Of course, we measure its velocity relative to a body outside. Thus, the principle of relativity can be alternatively stated as follows.

It is impossible to perform an experiment which will measure the state of uniform velocity of a system by observations, confined to that system.

The motion of a body itself has no meaning unless, we do not know with respect to which this motion has been measured. This led Newton to think about the absolute space and it represents an absolute frame with respect to which every motion should be measured. However, in view of this principle of relativity, we cannot perform an experiment which will measure the uniform velocity of a reference system relative to the absolute frame by observations confined to that system.

In the unaccelerated windowless ship all experiments performed inside it will appear the same whether this ship is stationary or in uniform motion. Newton stated the principle of relativity as follows:

The motions of bodies included in a given space are the same among themselves whether that space is at rest or moving uniformly forward in a straight line.

Study of the physical laws involves the measurements of accelerations, forces etc among bodies. The principle of relativity can be stated in an elegant form as follows:

The basic laws of physics are identical in all inertial systems which move with uniform velocity with respect to one another.

This principle is called **Galilean** or **Newtonian principle of relativity** and sometimes it is named as **hypothesis of Galilean invariance**. In fact, the principle of relativity is a fundamental postulate and is entirely consistent with the theory of special relativity. If any two inertial systems, moving with constant relative velocity, are connected by Galilean transformations, the principle of relativity is modified as:

The basic laws of physics are invariant in form in two reference systems connected by Galilean transformations.

This statement is somewhat special than the principle of relativity in the sense that it means the assumptions that the time and the space intervals are independent of the frame of reference. We shall see later in the theory of special relativity that the Galilean transformations are not correct, but the appropriate exact transformation equations are the Lorentz transformation equations for connecting any two frames in uniform relative motion. Thus, the principle of relativity may be stated as,

The basic laws of physics are invariant in form in two inertial frames connected by Lorentz transformations.

13.3 GALILEAN TRANSFORMATIONS:

At any instant, the coordinates of a point or particle in space will be different in different coordinate systems. The equations which provide the relationship between the coordinates of two reference systems are called **transformation equations**.

We have shown that a frame S' which is moving with constant velocity v relative to an inertial frame S, is itself inertial.

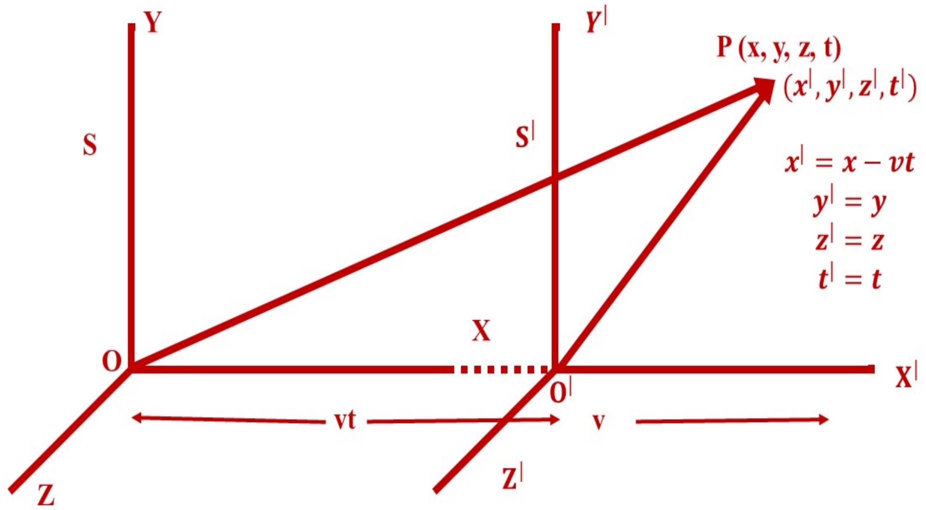

Fig. 13.1: Representation of Galilean Transformations

For convenience, if we assume (i) that the origins of the two frames coincide at t = 0, (ii) that the coordinate axes of the second frame are parallel to that of the first and (iii) that the velocity of the second frames are related by the equation frame relative to the first is v along X-axis, then the position vectors of a particle at any instant t in the two

$$r' = r - vt \qquad (1)$$

In the component form, the coordinates are related by the equations

$$x' = x - vt;\ y' = y;\ z' = z \qquad (2)$$

Eq. (1) or (2) expresses the transformation of coordinates from one inertial frame to another and they are referred as **Galilean transformations**.

The form of eq. (1) or (2) depends, of course, on the relative motion of two frames of reference, but it also depends upon certain assumptions regarding the nature of time and space. It is assumed that the time t is independent of any particular frame of reference i.e., if t and t' be the times recorded by the observers O and O' of an event occurring at P, then t' = t. If we add the equation t' = t, then the **Galilean transformation equations** are expressed as

$$x' = x - vt\ ;\ y' = y;\ z' = z;\ t' = t \qquad (3)$$

We can also consider that frame S is moving with velocity – v along the negative X-axis with respect to S' frame. Then the transformation equations from frame S' to S are

$$x = x' + vt',\ y = y'\ z = z'\ t = t' \qquad (3')$$

These are known as **inverse Galilean transformations**.

The other assumption, regarding the nature of the space, is that the distance between two points (or two particles) is independent of any particular frame of reference. Evidently, this assumption is expressed by the form of the transformation eq. (1) or (3). If a rod has length L in the frame S with the end coordinates x_1, and x_2, then $L = x_2 - x_1$.

If at the same time the end coordinates of the rod in S' are x_1' and x_2', then

$L' = x_2' - x_1'$. But for any time, t, from eq. (3), we have

$x_2' - x_1' = x_2 - x_1$

Therefore, $L' = L$ (4)

Thus, the length or distance between two points is invariant under Galilean transformations. Differentiating eq. (1) with respect to time, we get

$\frac{dr}{dt} = v + \frac{dr'}{dt} = v + \frac{dr'}{dt'}$ (or)

$u = v + u'$ .... (5)

where u and u' are the observed velocities in S and S' frames respectively.

Eq. (5) transforms the velocity of a particle from one frame to another and is known as **Galilean** (or **classical) law of addition of velocities**.

Again differentiating eq. (5) with respect to time t, we have

$\frac{du}{dt} = 0 + \frac{du'}{dt} = \frac{du'}{dt'}$ or a = a' ...... (6)

Hence according to Galilean transformations, the accelerations of a particle relative to S and S' frames are equal. It is to be mentioned that the Galilean transformations are based basically on two assumptions:

1) There exists a universal time t which is the same in all reference systems.
2) The distance between two points in various inertial systems is the same.

Thus, if any two events coincide for any observer, then they must occur simultaneously for all observers. In other words, the time interval between two given events must be identical for all systems reference.

13.4 SUMMARY:

This lesson introduces the foundational ideas of special relativity, starting with an understanding of Galilean transformations and the principle of relativity. The Galilean transformations describe how space and time coordinates transform when switching between inertial frames of reference in classical mechanics. However, the lesson emphasizes how the special theory of relativity, introduced by Einstein, alters these classical concepts, most notably by asserting that the speed of light is constant for all observers, regardless of their motion. This leads to the realization that time and space are not absolute but can vary depending on the observer's relative motion. By understanding these key concepts, students gain insight into how the laws of physics remain consistent across inertial frames and how classical mechanics is modified under high-speed conditions.

13.5 TECHNICAL TERMS:

Galilean transformations, Principle of relativity.

13.6 SELF-ASSESSED QUESTIONS:

1) What is principle of relativity? Explain
2) What do understand by the covariance of physical laws
3) How does the principle of relativity lead the constancy of speed of light in all inertial frames
4) Why is the speed of light significant in the context of the Principle of Relativity?
5) What is Galilean or Newtonian principle of relativity?

13.7 SUGGESTED READINGS:

1) Classical Mechanics by H. Goldstein.
2) Fundamentals of Classical Mechanics by J.C. Upadhyaya.
3) Classical Mechanics by G. Aruldhas, PHI Publishers.
4) The Theory of Relativity and Applications, Allen Rea.

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LESSON-14

THEORY OF RELATIVITY-II

14.0 AIM AND OBJECTIVES:

To learn about

- Transformation of force from one inertial system to another
- Covariance of the physical laws
- Principle of relativity and speed of light.

The aim of this lesson is to explore the fundamental concepts of the transformation of forces between inertial reference frames, understand the covariance of physical laws under these transformations, and grasp the principles of relativity, including the invariance of the speed of light in all inertial frames of reference.Learn how forces transform between different inertial reference frames using the principles of classical mechanics and special relativity.Investigate how the fundamental laws of physics remain unchanged or covariant when observed from different inertial frames.Understand that the laws of physics are the same in all inertial frames of reference, and how this leads to the relativity of simultaneity, time dilation, and length contraction.Study the concept that the speed of light is constant and the same in all inertial reference frames, irrespective of the motion of the source or observer.

STRUCTURE:

14.1 TRANSFORMATION OF FORCE FROM ONE INERTIAL SYSTEM TO ANOTHER:

Suppose that the force F on a particle of mass m in the frame S is represented by Newton's second law

$$\mathbf{F = ma} \qquad \textbf{(1)}$$

But according to the postulate that the laws of physics are the same in the frame S and in another frame S', which is in uniform motion relative to S, we have

$$\boldsymbol{F'} = \boldsymbol{m'a'} \qquad \textbf{(2)}$$

in the frame S'.

We have shown in the last article that the acceleration of the particle is the same in two inertial frames, connected by Galilean transformations, i.e.,

$$\boldsymbol{a'} = \boldsymbol{a} \qquad \textbf{(3)}$$

where in the deduction basically we have assumed the invariance of space and time separately.

In Newtonian mechanics, the mass is independent of velocity and hence

$$\boldsymbol{m} = \boldsymbol{m'}$$

Thus,

$$\boldsymbol{F} = \boldsymbol{ma} = \boldsymbol{m'a'} = \boldsymbol{F'} \qquad \textbf{(4)}$$

This means that if the relation F = ma (Newton's second law) is used to define the force, then inertial systems, the force F will have the same magnitude and direction, independent of the relative velocities of the reference frames. Further, the Newton's equation has the same form in the inertial frame S as well as in the frame S'. We mean this statement that Newton's second law is invariant under Galilean transformations. As Newton's first law (F = 0) can be deduced from second law and third law involves forces. We may also say that Newton's laws of motion (so called laws of mechanics) are invariant under Galilean transformations.

14.2 COVARIANCE OF THE PHYSICAL LAWS:

If the form of a law is not changed by certain coordinate transformation (i.e., if it is the same law in terms of either set of coordinates), we call that the law is invariant or covariant with respect to the coordinate transformation under consideration. Newton's laws of motion are covariant with respect to Galilean transformations. Mathematically, suppose a phenomenon is described in system S by an equation

$$\mathbf{f(x, y, z, t) = 0} \qquad \textbf{(5)}$$

Then the covariance of the equations means that in the system S', it will have the form

$$\boldsymbol{f(x', y', z', t')} = \mathbf{0} \qquad \textbf{(6)}$$

The principle of relativity asserts that the laws of physics are covariant in all inertial systems, moving with constant relative velocity. It is to be mentioned that the Galilean transformations satisfy the principle of relativity as far as Newton's laws of motion are concerned, but as we shall see later, these transformations do not satisfy this principle for propagation of electromagnetic waves.

14.3 PRINCIPLE OF RELATIVITY AND SPEED OF LIGHT:

According to the principle of relativity, basic laws of physics remain the same in all inertial systems. If the principle of relativity is extended to electrodynamics, Maxwell's fundamental equations should remain the same in any two inertial systems with uniform constant relative motion. It follows from Maxwell's equations that the electromagnetic waves are propagated in vacuum with a constant velocity c = 3 x 10^8 m/sec in all directions irrespective of the motion of the source. Light waves are basically electromagnetic waves and hence according to the principle of relativity, the velocity of light must be the same with value c in all inertial systems, independent of the motion of the light source.

It can be shown that the idea of constancy of speed of light contradicts the Galilean transformations. Let S be frame of reference with a source of light at the origin O. In this system, the velocity of light is c in all the directions. Now, let a frame S' be moving with constant velocity $v = v\hat{i}$ along X-axis. In the frame S', the velocity of light, using Galilean transformations, will be given by

$$\boldsymbol{c' = c - v} \qquad \textbf{(7)}$$

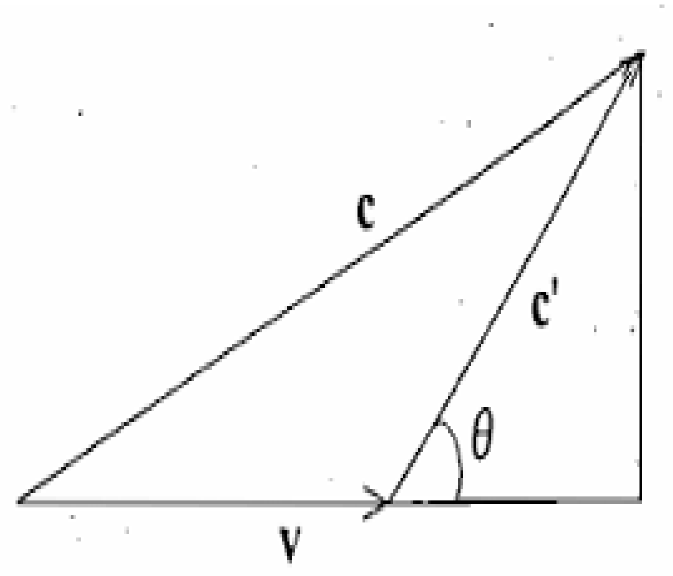

Fig. 14.1

The speed of the light signal along X-axis (θ = 0) will be noted in S' as

$$\boldsymbol{c' = c - v} \quad \textbf{(8)}$$

and along Y'-axis ($\theta = \pi/2$) as

$$\boldsymbol{c' = \sqrt{c^2 - v^2}} \ \ \textbf{(as } \boldsymbol{c^2 = c'^2 + v^2}\textbf{)} \quad \textbf{(9)}$$

Hence, if we use Galilean transformations, we find that the speed of light is not constant in all inertial systems and this contradicts the principle of relativity. Further S must be a **preferred** or **absolute** frame in which the speed of light is c and hence any other inertial frame (S') should be less suitable. This leads the possibility of defining **absolute motion**. If we accept the principle of relativity in the fields of electromagnetism and optics, we should revise the concepts of space and time. However, it seems necessary that before adopting a radical departure from the classical ideas of space and time, one should be sure for the truth of the new step by experiments. Michelson Morley experiments were performed to detect the influence of the motion of the earth with respect to so called absolute frame. Negative results were obtained from these experiments and this led finally the acceptance of the principle of relativity.

14.5 SUMMARY:

This lesson delves into the core principles of relativity and the transformation of forces between different inertial reference frames. The transformation equations illustrate how forces behave in moving frames of reference, linking classical mechanics to relativistic effects. It emphasizes the covariance of physical laws, showing that the fundamental laws governing nature remain consistent across inertial frames. The principle of relativity is discussed, highlighting that no preferred frame exists and the laws of physics are invariant across all inertial frames. Lastly, the constancy of the speed of light is examined, reinforcing one of the cornerstones of modern physics, which leads to profound implications for time, space, and motion at high velocities.

14.6 TECHNICAL TERMS:

Transformation of force, covariance, principle of relativity and speed of light.

14.7 SELF-ASSESSMENT QUESTIONS:

1) Write a note on the Transformation of force from one inertial system to another.
2) Discuss the covariance of the physical laws.
3) Write the principle of relativity and speed of light.

14.8 SUGGESTED READINGS:

1) Classical Mechanics by H.Goldstein.

2) Fundamentals of Classical Mechanics by J.C. Upadhyaya.

3) Classical Mechanics by G. Aruldhas, PHI Publishers.

4) The Theory of relativity and applications, Allen Rea.

Prof. G. Naga Raju

LESSON-15

APPLICATIONS OF SPECIAL THEORY OF RELATIVITY

15.0 AIM AND OBJECTIVES:

To learn about-

- Lorentz Transformations.
- Consequences of Lorentz Transformations.

STRUCTURE:

15.1 INTRODUCTION:

The chapter on the theory of relativity introduces the foundational principles of Lorentz transformations, which describe how measurements of space and time change for observers in different inertial frames. The Lorentz transformations are crucial for understanding relativistic effects that emerge at high velocities, approaching the speed of light. This chapter explores the consequences of Lorentz transformations, including phenomena such as aberration of light from stars, length contraction, time dilation, and the energy-mass relation. These concepts challenge our classical intuitions about time and space, highlighting the interconnectedness of physical realities in the framework of modern physics.

15.2 LORENTZ TRANSFORMATIONS:

From practical point of view at low speeds, there is no difference between the Lorentzian and Galilean transformations and we use the later in most of the problems which we encounter. However, when we have to deal with very fast particles having velocities comparable to c, such as electrons in the atoms, cosmic ray particles, we must use the Lorentz transformations.

Suppose that S and S' be the two inertial frames of reference. S' is moving along positive direction of X-axis with velocity v relative to the frame S. Let t and t' be the times recorded in two frames. For our convenience, we will assume that the origins O and O' of the two co-ordinate systems coincide at t = t' = 0.

Now suppose that a source of light is situated at the origin O in the frame S, from which a wavefront of light is emitted at t = 0. When the light reaches at the point P, let the positions and times, measured by the observers O and O', be (x, y, z, t) and (x', y', z', t') respectively. If the velocity of light is c, then the time measured by the light signal in traversing the distance OP in frame S is

$$t = \frac{OP}{C} = \frac{(x^2+y^2+z^2)^{1/2}}{c} \text{ or } x^2+y^2+z^2 = c^2t^2 \qquad (1)$$

This equation represents the equation of wavefront in frame S. According to the special theory of relativity, the velocity of light will be c in the second frame S'. Hence, in frame S', the time required by the light signal in travelling the distance O'P is given by

$$x' = \frac{O'P}{C} = \frac{(x'^2+y'^2+z'^2)^{1/2}}{c} \text{ or } x'^2+y'^2+z'^2 = c^2t^2 \qquad (2)$$

which is the equation of the wavefront in frame S'.

Now transformation equations relating x, y, z, t and x', y', z', t' should be such that eq. (2) transforms to ea. (1). The Galilean transformations connect the measurements in the two frames according to the following equations:

x' = x - vt ; y' = y; z' = z; t' =t

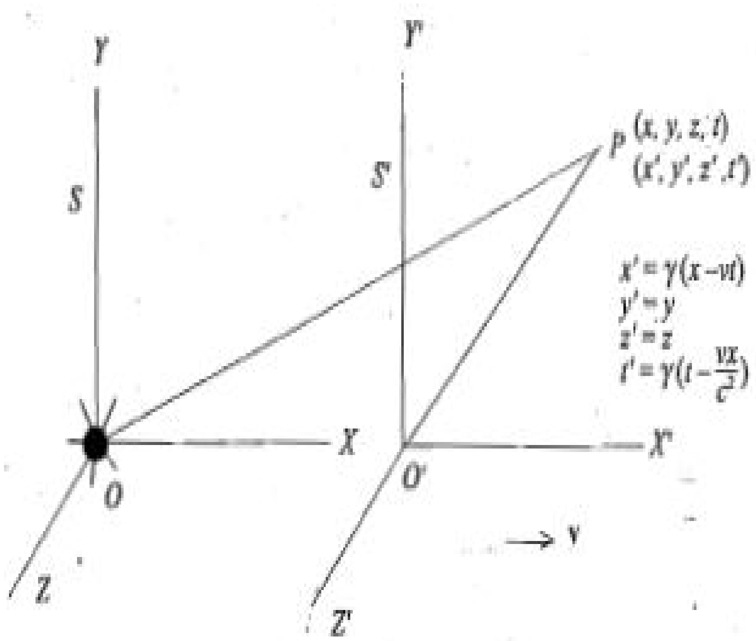

Fig.15.1: Representation of Lorentz Transformations

Substituting these values in eq. (2), we get

$(x-vt)^2+y^2+z^2-c^2t^2 = 0$ or

$x^2-2xvt+v^2t^2+y^2+z^2-c^2t^2 = 0$ (3)

This equation is certainly not in agreement with eq.(1) because it contains an extra term $(-2xvt+v^2t^2)$.Thus the Galilean transformation fails. Further t + t'. (because t=OP/c and t'=O'P/c) which does not agree with the Galilean transformation equations. If the principle of the constancy of the speed of light is valid in all frames, there should exists some transformation equations which reduces to the Galilean one for v/c→0 and which transform $x'^2+y'^2+z'^2-c^2t^2=0$ into $x^2+y^2+z^2-c^2t^2=0$

When we look at the eq. (1) and eq. (3), we find that the terms of y and z are in agreement. Hence we can say y'= y and z'= z. The extra term $(-2xvt+v^2t^2)$ indicates that transformation in x and t should be modified so that this extra term is cancelled.

We note that for the observer O, the distance OO' = vt and therefore when x' = 0 (point *O*'), x=vt. This suggests the transformation x'= α(x-vt) because only for x'=0, x=vt. Since t' is different from *t* and may be depending on x, so that in general we may also assume that t' = α' *(t* + fx). Here a, a' and *f* are constants, to be determined (for Galilean transformations ar a'=1 and *f*=0). Now substitution for x', y', z' and *t*' in (11), we have

$\propto^2 (x-vt)^2 + y^2 + z^2 = c^2 \propto^2 (t+fx)^2$ Or

$$\propto^2 (x^2 - 2vxt + v^2t^2) + y^2 + z^2 = c^2 \propto^2 (t^2 + 2fxt + f^2x^2)$$

or $x^2(\propto^2 - f^2\alpha'^2c^2)$-2x$(\propto^2 v + f\alpha'^2c^2) + y^2 + z^2) =$

$\left(\alpha'^2 - \frac{\propto^2 v^2}{c^2}\right) c^2t^2$ (4)

This result obtained from applying transformations from S' to S, must be identical to eq. (1). Therefore,

$\left(\propto^2 - f^2\alpha'^2c^2\right) = 1$ (i)

$(\propto^2 v + f\alpha'^2c^2) = 0$ (ii)

$\left(\alpha'^2 - \frac{\propto^2 v^2}{c^2}\right) = 1$ (iii)

Substituting the value of $f = \frac{\alpha^2 v^2}{\alpha'^2 c^2}$ from (ii) in eq.(i), we get

$\alpha^2 - \frac{\alpha^4 v^2}{c^2} = 1$ or $1 - \frac{\alpha^2 v^2}{\alpha'^2 c^2} = \frac{1}{\alpha^2}$

But from (iii) $\alpha'^2 = 1 + \frac{\alpha^2 v^2}{c^2}$, hence

$1 - \frac{\alpha^2 v^2/c^2}{1+\alpha^2 v^2/c^2} = \frac{1}{\alpha^2}$ or $\frac{1}{1+\alpha^2 v^2/c^2} = \frac{1}{\alpha^2}$ or $1 + \frac{\alpha^2 v^2}{c^2}$

or $\alpha^2 = \frac{1}{1-v^2/c^2}$ or $\alpha = \frac{1}{\sqrt{1-v^2/c^2}}$

Therefore,

$\alpha'^2 = 1 + \frac{v^2/c^2}{1-v^2/c^2} = \frac{1}{1-v^2/c^2}$ or $\alpha' = \frac{1}{\sqrt{1-v^2/c^2}}$

Thus from (ii) $f = -\frac{v}{c^2}$

Therefore,

$$x' = \alpha\,(x - vt) = (x - vt)/\sqrt{1 - v^2/c^2} \text{ and } t' = \alpha'(t+fx) = (t - {}^{vx}/_{c^2})/\sqrt{1 - v^2/c^2}$$

Thus, the new transformation equations, which are in agreement with the invariance of velocity of light c, are

$$x' = \frac{(x-vt)}{\sqrt{1-v^2/c^2}}\,;\ y' = y\,, z' = z\,;\ t' = \frac{(t - {}^{vx}/_{c^2})}{\sqrt{1-v^2/c^2}} \qquad (5)$$

These equations are called Lorentz transformations, because they were first obtained by Dutch physicist H. Lorentz.

We note that when v «c i.e, we get the Galilean transformations from the Lorentz transformations. In most of the cases, which we encounter on earth, < is a velocity very large compared with the great majority of velocities i.e., v<< c so that the results of Lorentz transformations do not to any great extent from those of the Galilean transformations; but from a theoretical point of view the Lorentz transformations represent a most profound conceptual change specially in relation to space and time.

For convenience, sometimes we put $\beta = v/c$ and $1/\sqrt{1 - v^2/c^2} = 1/\sqrt{1 - \beta^2} = \gamma$

Hence the transformations are written as

$$x' = y(x - vt);\ y' = y; z' = z; t' = \gamma(t - \frac{vx}{c^2}) \qquad (6)$$

In the derivation of these equations, we assumed that frame S' is moving in positive X-direction with velocity v relative to the frame S. But if we say that frame S is moving with –v velocity relative to S' along negative X-direction, then the transformations are

$$x = \gamma(x' + vt);$$

$$y = y'; \ z = z'$$

$$t = \gamma\left(t' + \frac{vx'}{c^2}\right) \quad (7)$$

These are known as inverse

15.3 CONSEQUENCES OF LORENTZ TRANSFORMATIONS:

15.3.1 Aberration of Light from Stars:

By "aberration" I am not referring to optical aberrations produced by lenses and mirrors, such as coma and astigmatism and similar optical aberrations, but rather to the *aberration of light* resulting from the vector difference between the velocity of light and the velocity of Earth.

The effect of aberration is to displace a star towards the *Apex of the Earth's Way*, which is the point on the celestial sphere towards which Earth is moving. The apex is where the ecliptic intersects the observer's meridian at 6 hours local apparent solar time. The amount of the aberrational displacement varies with position on the sky, being greatest for stars 90°90° from the apex. It is then of magnitude v/cv/c, where v v and cc are the speeds of Earth and light respectively. This amounts to 20.5 arc seconds. (You didn't know that the speed of Earth could be expressed in arc seconds, did you?) But what matters in astrometry is the *differential aberration* between one edge of the detector (photographic film or CCD) and the other. This is going to be a much smaller effect than differential refraction.

Let us examine the effect of aberration in figures:

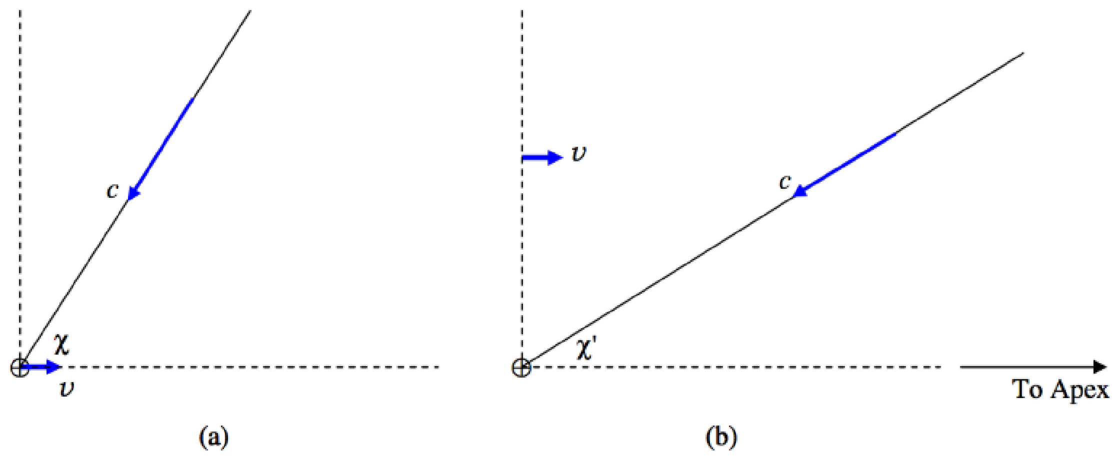

Fig. 15.2 Effect of Aberration

Part (a) of the figure shows a stationary reference frame. By “stationary” I mean a frame in which Earth, is moving towards the apex at speed v=29.8 km s−1v=29.8 km s^{-1}. Light from a star is approaching Earth at speed cc from a direction that makes an angle χ, which shall call the true apical distance, with the direction to the apex.

Part (b) shows the same situation referred to a frame in which Earth is stationary; that is the frame (b) is moving towards the apex with speed vv relative to the frame (a). Referred to this frame, the speed of light is cc, and it is coming from a direction $\chi'\chi'$, which I shall call the *apparent apical distance*.

The difference $\varepsilon=\chi - \chi'$ as the *aberrational displacement.*

For brevity I shall refer to the direction to the apex as the “x-direction” and the upwards direction in the figures as the “y-direction”.

Referred to frame (a), the x-component of the velocity of light is $-c\cos\chi$, referred to frame (b), the xx-component of the velocity of light is $-c\cos\chi'$. These are related by the Lorentz transformation between velocity components:

$$c\cos\chi'=c\cos\chi+v/1+(v/c)\cos\chi. \qquad (8)$$

Referred to frame (b), the y-component of the velocity of light is $-c\sin\chi$ and referred to frame (b), the y-component of the velocity of light is $-c\sin\chi'$. These are related by the Lorentz transformation between velocity components:

$$c\sin\chi'=c\sin\chi\gamma(1+(v/c)\cos\chi), \qquad (9)$$

in which, if need be, a cc can be canceled from each side of the Equation. In Equation (9),

γ is the Lorentz factor $\sqrt{1-\frac{v^2}{c^2}}$.

Equations (8) and (9) are not independent; indeed one may be regarded as just another way of writing the other. One easy way to show this, for example, is to show that $\sin^2\chi'+\cos^2\chi'=1$.

15.3.2 Length Contraction:

Having seen that time interval measurements in two reference frames are different, it is natural to expect the same about length measurements too. The definition of proper length goes along the same line as that of proper time. Definition: The Proper length of an object is its length measured in a frame in which the object is at rest. Let us compare the length of an object as measured in two reference frames. Frame S in which the object is at rest with length L, say along the x direction, and another frame S ′ in motion with respect to S along the x direction with speed v. The world lines of the object in the two frames are shown below. Let A and B be the two events at which the two ends of the object cross the observer. In S

$A = (ct_A; x_A); B = (ct_B; x_B)$ (10)

and in S ′

$A = (ct'_A; 0) ; B = (ct'_B; 0).$ (11)

Then the length of the object in S′ is given by

$L' = v(t'_B - t'_A).$ (12)

The space time interval between A and B is the same in the two reference frames. This gives

$c^2(t'_B - t'_A)^2 = c^2 (t_B - t_A)^2 - (x_B - x_A)^2$

$\Rightarrow c^2L'^2/ v^2 = c^2/v^2(L^2 - L^2)$ (13)

or,

$L' = L\sqrt{1-\frac{v^2}{c^2}} = L/\gamma .$ (14)

Since $\gamma \geq 1$, $L' \leq L$, i.e. objects in motion appear smaller along the direction of motion.

PS: We have considered only the contraction along the direction of motion. However, it is straightforward to show (using detailed space-time diagrams) that in the other two directions, i.e., the direction perpendicular to the motion, there is no contraction.

15.3.3. Time Dilation:

The Proper time between two events is defined as the time interval between the events in a frame in which the two events happen at the same place. The time shown by a clock in a reference frame in which the clock is at rest is the Proper time shown by the clock. Evidently, proper time is not defined between two events that are space like separated. The most important facet of special relativity is that it identifies space and time not as separate entities but part of a single space-time continuum. It forces us a rethink on our basic notion of time and space measurements in the elementary of situations. Consider a clock at rest in a reference frame S. The world line of the clock is shown in Figure

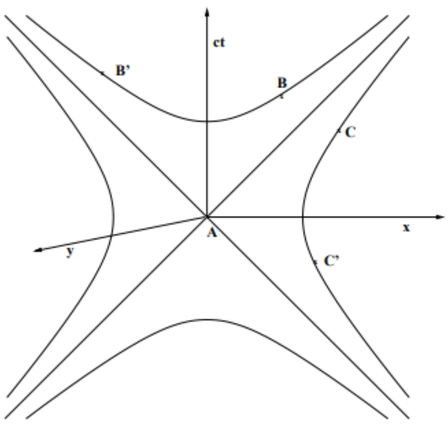

Fig: Event B might transform to B′ in some other reference frame S ′. However, the time ordering between events A and B is preserved (A′ = A). This is true for any two events that are time like separated.

But the same is not true for events C and A, which are space like separated. C transforms to C′ in S′. Since the time ordering is reversed, A precedes C in S, but is reversed in S′ . The ticking of the clock at intervals t_1, t_2,.. are the events denoted by the dots along the time axis. The space-time interval between the first two ’ticks’ is given by

$$\Delta s^2 = c^2 (t_2 - t_1)^2 \qquad (15)$$

Let S ′ be a reference frame moving with speed v in the x direction with respect to S. The two ticks of the clock in this reference frame do not happen at the same point in space. Let the two events in this frame be (ct'_1, x'_1) and (ct'_2, x'_2) (we do not bother about the y ′,z′ coordinates since they are same as the unprimed ones, the motion being in the x direction). The space time interval in this frame is given by

$$\Delta s'^2 = c^2 (t'_2 - t'_1)^2 - (x'_2 - x'_1)^2 \qquad (16)$$

$$= c^2(t'_2 - t'_1)^2 - v^2(t'_2 - t'_1)^2 \qquad (17)$$

Since the two intervals are same by Postulate II, it follows that

$$(t_2 - t_1) = \sqrt{1 - \frac{v^2}{c^2}} \cdot (t'_2 - t'_1) \qquad (18)$$

Or

$$(t'_2 - t'_1) = \gamma\, (t_2 - t_1) \qquad (19)$$

where $\gamma = \sqrt{1 - \frac{v^2}{c^2}}$. So, the times shown by a moving clock will be different from the one it shows when it is at rest. It can be seen that $1 \le \gamma \le \infty$. $(t'_2 - t'_1)$ is the time interval between two ’ticks’ in a frame in which the clock is moving. From Eq. (19) it follows that $(t'_2 - t'_1) \ge (t_2 - t_1)$. Moving clocks run slower by a factor γ.

14.3.4 Energy-Mass Relation:

Suppose a force $F = \frac{d}{dt}(mv)$ be acting on a particle of mass m so that its kinetic energy increases. The gain in kinetic energy will be equal to the work done on the particle. If the force displaces the particle through a distance dr along its line of action, then the infinitesimal gain in the kinetic energy is

$$dE_k = Fdr = \frac{d}{dt}(mv)dr = vd(mv) \quad \text{(Because } v = \frac{dr}{dt}\text{)}$$

If the particle starts from rest (v = 0) and acquires velocity v under the action of the force, then the gain in the kinetic energy by the particle will be given by

$$E_k = \int d\,E_k = \int_0^v vd(mv)$$

Integrating this equation by parts, we obtain

$E_k = vmv \vdots \int_0^v mv - \int_0^v mvdv = mv^2 - \int_0^v \frac{m_o vdv}{\sqrt{1-v^2/c^2}}$

$= \frac{m_o v^2}{\sqrt{1-v^2/c^2}} + m_0 c^2 \sqrt{1-\frac{v^2}{c^2}} - m_0 c^2 = \frac{m_o c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2 = mc^2 - m_0 c^2$

Thus,

$E_k = (m - m_0)c^2 = \frac{m_o c^2}{\sqrt{1-v^2/c^2}} - m_0 c^2$ .... (20)

Where $m = m_0 / \sqrt{1-v^2/c^2}$

Eq. (20) is the expression for relativistic kinetic energy. It shows that *the gain in kinetic energy corresponds to an increase in mass.*

The quantity $m_o c^2$, occurring in relation (20), is due to the rest mass of the particle and is called the *rest energy* or *proper energy* E_0, of the particle, *i.e.,* $E_0 = m_0 c^2$.Thus, the *total energy* of the particle, when it is moving with velocity v, is

E = kinetic energy (E_k) + Rest energy (E_0)

= $(m - m_0)c^2 + m_0 c^2 = mc^2$

Thus E=$\frac{m_o c^2}{\sqrt{1-v^2/c^2}} = mc^2$ ...... (21)

For integrating put $1 - v^2/c^2 = \alpha$ and hence $v\,dv = -c^2 d\alpha/2$

This energy *E* is called the *relativistic energy* (total energy) of a particle, having relativistic mass *m.* Thus, there exists a very close relation between mass and energy, unknown in classical physics. This is well known Einstein's mass-energy relation.

The relativistic kinetic energy can be expressed as

$$E_k = E - E_0 = (m - m_0)c^2 = m_0 c^2 (1 - v^2/c^2)^{-1/2} - m_0 c^2$$

$E_k = m_0 c^2 [\,1 + \frac{1}{2}.\frac{v^2}{c^2} + \frac{3}{8}.\frac{v^4}{c^4}$ +......]- $m_0 c^2$ (Using Bionomial theorem)

In the limit we have v^2/c^2 $less\ than << 1$, $we\ have$

$E_k = m_0 c^2 \left(1 + \frac{v^2}{2c^2}\right) - m_0 c^2 = \frac{1}{2} m_0 v^2$..... (22)

This relation is the classical result for the kinetic energy.

We see from eq. (20) and (21) that the increase in kinetic energy or total energy ΔE of a particle is associated with a corresponding increase in mass Δm according to the relation,

$\Delta E = \Delta mc^2 \ldots\ldots$ (23)

It is known that one kind of energy, e.g., kinetic energy can be converted in other forms and hence all forms of energy must be associated with them some mass. According to Einstein, eq. (23) is the most important consequence of the special theory of relativity. He considers that an amount of energy ΔEin any form is equivalent to a mass Δm = $\Delta E/c^2$ and conversely, any mass Δm is equivalent to an energy

$$\Delta E = \Delta mc^2$$

This is called the *principle of equivalence of mass and energy*. Thus there is the possibility that mass can be changed into energy and vice-versa. The truth of this fact has been verified by a number of experiments.

In the language of Einstein, the mass of a body is the measure of the quantity of its energy. This means that a system of inertial mass m is equivalent to an energy $E = mc^2$. Further the rest mass of a body cannot be distinguished from the mass due to the energy possessed by it. Thus, there is the possibility that the rest mass of a body is due to some form of energy and an interchange between rest mass and energy may occur.

15.4 SUMMARY:

In this chapter, the implications of Lorentz transformations are examined in detail. The aberration of light reveals how the motion of the observer influences the perceived position of stars. Length contraction illustrates how objects appear shorter when in motion relative to an observer. Time dilation demonstrates that time can flow at different rates depending on relative velocities. Finally, the energy-mass relation, epitomized by the equation $E=mc^2$, succinctly captures the equivalence of mass and energy. These consequences fundamentally alter our understanding of the universe, showcasing the profound effects of relativity on both theoretical physics and practical applications.

15.5 TECHNICAL TERMS:

Lorentz Transformations, Aberration of Light from Stars and Time dilation.

15.6 SELF-ASSESSED QUESTIONS:

1) Lorentz transformation is the relationship between two different coordinate frames that move at a constant velocity and are relative to each other.

2) Time dilation is the difference in elapsed time between two clocks that are in motion relative to each other. Time dilation is also the difference in elapsed time between two clocks that are experiencing gravitational fields of different magnitudes.
3) Explain the significance of Lorentz transformations in the context of theory of relativity. How do they alter our understanding of space and time?
4) Describe the aberration of light from stars. How does this effect provide evidence for the theory of relativity?
5) Derive the expressions for Lorentz space time transformation
6) What are Lorentz transformations, and why are they essential in relativity?
7) How does the aberration of light affect our observations of celestial bodies?
8) Define length contraction and provide an example of its significance.
9) What is time dilation, and how does it impact the perception of time for moving observers?
10) State the energy-mass relation and explain its importance in modern physics?

15.7 SUGGESTED READINGS:

1) Classical Mechanics by H.Goldstein.
2) Fundamentals of Classical Mechanics by J.C. Upadhyaya.
3) Classical Mechanics by G. Aruldhas, PHI Publishers.
4) The Theory of relativity and applications, Allen Rea.

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